A TREATISE

ON

STATICS,

CONTAINING THE

THEORY OF THE EQUILIBRIUM OF FORCES;

AND

Numerous Examples

ILLUSTRATIVE OF THE GENERAL PRINCIPLES OF THE SCIENCE.

BY

S. EARNSHAW, M. A.
OF ST JOHN'S COLLEGE, CAMBRIDGE.

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PREFACE TO THE THIRD EDITION.

THIS edition of the Treatise on Statics differs so little from the last as scarcely to require a separate notice. A few Articles have been added, on the pressure which a rigid body is made to exert on a fixed point or axis of support by the action of forces when there is equilibrium. These will be found useful in those Problems of Dynamics wherein it is required to find the pressure which a rigid body in motion, under the influence of any forces, exerts on a fixed point or axis. Indeed, it is chiefly with a view to this application of them that the articles alluded to have been introduced into this edition of the Statics.

The collection of Problems for practice given at the end of the Treatise has been considerably enlarged, chiefly by the addition of Examples of an elementary character. In the selection of them care has been taken to choose such as illustrate Statical Principles under every important variation of aspect, without impeding the student's progress through them by analytical and other difficulties foreign to the proper object of this Treatise.

CAMBRIDGE, *Feb.* 1, 1845.

PREFACE TO THE SECOND EDITION.

THOUGH the general plan and arrangement of this edition of the Treatise on Statics are the same as in the former, in the details there will be found, it is hoped, some important improvements.

The fundamental proposition of the science,—the Parallelogram of Forces,—I have proved after Duchayla's method, by reason of its simplicity; but I think it necessary here to inform the reader that, as that method is inapplicable when the forces act upon a single particle of matter (as a particle of a fluid medium on the hypothesis of finite intervals), on account of its assuming the transmissibility of the forces to other points than that on which they act, I have, in an Appendix, given the proof which in the first edition was given in the text. The same objection (and for the same reason) lies against the proof of the parallelogram of forces from the properties of the lever. This method, though allowable in the infancy of the science, can never be exclusively adopted in a Treatise which professes to take a more philosophical view of the subject; for, were the transmissibility of force *not* true in fact, the law of the composition of forces acting on a point would still be true; it is evident, therefore, that to make the truth of the former an essential step in the proof of the latter, is erroneous in principle.

In the former edition, forces were considered as acting in any directions in space; a mode of treatment of the subject which necessarily rendered the investigations useless to such readers as had not studied Geometry of Three Dimensions. In the present edition this defect is remedied; and a chapter, in which the forces are supposed to act in a plane, is always made to precede the more general investigations. At the end of Chapter IV. several propositions are proved which have hitherto been used in Elementary Books without proof.

The fifth Chapter contains a new (and it is hoped a satisfactory) and complete proof of the Principle of Virtual Velocities, and its Converse. The proof given by Lagrange in his *Mécanique Analytique*, page 22, et seq., though highly ingenious, I regard as a fallacy; and, if not fallacious, deficient in generality.

In the last Chapter, I have endeavoured to set before the reader such problems as, without involving analytical difficulties, seemed best calculated to make him acquainted with the mode of applying all the most important principles of the science: and not unfrequently I have added remarks upon important steps with the view of pressing them more particularly upon the reader's attention.

ST JOHN'S COLLEGE, CAMBRIDGE,
March 12, 1842.

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By the same Author.

A TREATISE ON DYNAMICS. *Third Edition.*

STATICS.

INTRODUCTION.

DEFINITIONS AND PRELIMINARY NOTIONS.

1. IN the Science of MECHANICS of which STATICS forms a part, *matter* is considered as essentially possessing *extension*, *figure* and *impenetrability*. The least conceivable portion of matter is called a *particle*.

2. We conceive of matter that it can exist either in a state of *rest*, or *motion*. If then matter, once at rest, pass into a state of motion, the change, not being essential to the existence or nature of matter, is of necessity ascribed to some agent, which, as to its nature, is essentially independent of the matter influenced. Whether this agent reside in the matter influenced, or in external objects, or in both, are questions which can only be answered after experimental investigation. This agent is called *force*; and it will be perceived from this statement, that a force is judged of entirely by the *effects* which it produces: and hence, if in the same circumstances two forces produce equal effects, we infer that the forces are equal.

3. It is *assumed*, that the effect of two equal forces acting in concert, is double the effect of one of them; three, treble; and so on.

The reason of its being necessary to make this an assumption is, that in our ignorance of the nature of force, we are compelled to judge of it by the *change* which it produces in the state of rest or motion of matter; and it is obvious, that we can no more judge that one such *change* is twice as great as another, than we can affirm that one candle is twice as bright, or one substance twice as sweet, or one noise twice as loud as another.

4. A force is considered as having *magnitude* and *direction*, and a *point of application*. When these three are known, the force is said to be known. From Art. 2, it will be seen that, by the magnitude of a force, we mean the degree of motion which it is capable of producing in matter previously at rest; and by the direction of a force, we mean the direction in which a particle of matter, under the influence of that force, would begin to move; and by the point of application of a force, we mean that particular particle of a mass of matter on which the force immediately exerts its influence.

5. If one particle of a rigid* mass of matter be acted upon by a force, it cannot obey the influence of the force without dragging with it the other matter with which it is connected; the motion therefore which it would receive, if free, is in some manner distributed among the whole mass of which it is a part. It is clear, therefore, that the subject of which we are treating, naturally divides itself into two distinct parts, according as the forces act on a free particle, or on a rigid body.

6. With respect to the motion of a particle of matter, we conceive that it consists in the particle's being found to occupy different parts of space at successive instants, or epochs of time; but with respect to the motion of a rigid body we conceive,

(1) That as a *whole* it may occupy the same portion of space at successive epochs, while some of its parts individually occupy different parts of space in successive instants.

This is called *rotatory* motion.

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* We define a rigid body to be an assemblage of particles of matter, connected together in such a manner that their relative places never change.

(2) That as a whole it may occupy different parts of space at successive epochs, without having at the same time any rotatory motion.

This is called a motion of translation.

(3) That both these kinds of motion may exist together in the same body.

This is the most general kind of motion of which we can form a notion.

7. From the preceding articles it will be perceived that we have taken motion as the characteristic effect of force. It will now be necessary to shew, that there exists another effect (and that more convenient for our present purpose) which may be taken as the measure or characteristic of force.

If any portion of matter (a stone for instance) be held in the hand, it will be found to exert a pressure; and if the hand be suddenly removed, will fall. In its fall it may be caught, but the hand will again feel a pressure. This experiment informs us, that that which is the cause of motion, is likewise the cause of pressure. While the stone is held at rest, its continual tendency to fall is evidenced by the pressure which is exerted on the hand; hence, in all cases where motion is prevented, there is pressure. But further, the latter part of the experiment teaches us that, in all cases where motion is retarded, there is pressure. If when the stone is at rest, the hand exert a greater pressure upwards than is necessary to prevent it from falling, the stone will begin to move upwards. Hence we learn that pressure attends the production as well as the prevention and the destruction of motion. Thus it appears that pressure produces the same results as we have taken to be the characteristic effects of force. We may therefore take pressure as the measure of force, because both pressure and motion are effects of the same cause.

8. The Earth, in some unseen manner, exerts a pressure in a downwards direction upon all matter with which we are

acquainted. This pressure it is which occasions the descent of falling bodies to the ground, and causes all bodies lying on the ground to press against it. More accurate experiments prove that *every* particle of matter, whether of metal, wood, earth, or of any other substance, is subject to this influence. And it can be shewn that the degree of this pressure exerted upon a given body never changes. Thus, let a spring AB have one end A firmly fixed in an immoveable block. Suspend a proposed substance P from the other end B, then the spring will be bent in the manner represented in fig. 1, the point B taking a position B'. If the experiment be again tried with the same body P after any interval of time, it will be found that the spring will be bent exactly as at first; thus shewing that the Earth exerts an unvarying pressure upon every body.

If the experiment be tried with the same spring and substance P at a place in another latitude, or on a hill, or in a pit, the bending of the spring is not found to be the same as before: but at the same place no variation is ever observed in the result.

9. We may easily find other substances P', P'', P'''... each of which being suspended from B will bend the spring exactly as P does. By suspending 2, 3, 4, ... of these bodies at a time, and marking the spaces through which the spring is bent in each case, we may form a graduated scale, by means of which we can ascertain exactly the degree of pressure which the Earth exerts upon any proposed body whatever, as compared with the pressure which it exerts upon P. If this be done, it is usual to call the pressure on P the unit of pressure; and the pressure which is exerted upon another body, if it be W times the pressure on P, is said to be equal to W.

10. The pressure W which the *Earth* exerts upon a body, when measured in the manner just described, is called the *weight* of the body. How great soever be the pressure which any other force exerts upon a body, we can always find (hypothetically at least) so many bodies P, P', P'', P'''... that the

Earth shall exert upon them, taken all together, exactly as much pressure as the proposed force exerts upon the proposed body. Hence then, with the assumption in Art. 3, we perceive that every force may be measured, and therefore represented, by a weight.

11. To avoid circumlocution, when a body is prevented by an obstacle from moving, it is usual to say that the *body* exerts a pressure upon the obstacle, and that the obstacle exerts an equal pressure upon the body in the contrary direction. The fact however is, that the body is completely passive; and the reason why it remains in a state of rest is, that it is under the influence of two equal pressures exerted on it in opposite directions. By the same licence, if a body, which is under the influence of the Earth's action, be suspended by a string, it is often said that the string exerts a force or pressure upon the body; the fact however in this case is, that the string by being attached to the body, becomes a part of the body; and the whole remains in a state of rest, for the same reason as before. Hence it will be seen that, in the experiment described in Art. 8, the spring exerts a force equal to that exerted by the Earth upon P, though in the contrary direction. And hence we say, when two bodies are pressed together, that they act and react upon each other with equal forces.

12. It is sufficiently evident, that two equal pressures, acting in opposite directions upon the same point of a body, counteract each other: but it is conceivable that if several pressures be applied to a body, even though they be not two and two in opposite directions, nor all applied to the same point of the body, they may counteract each other. The science which teaches the relations necessarily existing between the *magnitudes* of forces, their *directions*, and their *points of application*, when they exactly counteract each other, is called STATICS.

13. If several forces acting upon a body counteract each other, the body is said to be in *equilibrium:* and the forces are said to *balance* each other.

14. If several forces acting upon a free particle do not balance each other, the particle will begin to move in some direction in a certain manner. It may be prevented from so moving by applying a proper force in the opposite direction to that in which there is a tendency to motion. This new force exactly counteracts the whole system of forces: but it might be itself counteracted by a single force equal to itself and acting in a contrary direction. A single force satisfying these conditions would be exactly equivalent to the whole of the original system of forces; and it is therefore called their *resultant*.

15. We thus learn that several forces, if they act upon a free particle, may be replaced by one force; and the converse is evidently true, viz. that one force may be replaced by a system of several forces. When one force is replaced by a system of several forces, they are called its *components*.

16. By reference to Art. 6, we see that the motions which a rigid body may take are of two distinct kinds: and therefore the reasoning just stated respecting a free particle does not apply to rigid bodies. We shall hereafter shew that, corresponding to the three cases stated in the Article referred to, a system of forces acting on a rigid body may have

(1) A resultant for rotation only,

(2) A resultant for translation only,

(3) Two resultants, one for the rotation and one for the translation.

17. It is evident from the explanations above given, that a system of forces, acting on a free particle, cannot have more than one resultant: but we have just seen that the same is not necessarily true when they act on a rigid body. It is always true, however, that the same force may have different systems of components.

18. If a particle, or a rigid body, be in equilibrium under the action of several forces, we may add to the system, or take away from it, any set of forces which balance each other.

This principle is called the " superposition of equilibrium," and we shall hereafter have frequent instances of its utility.

19. It follows at once from this, that when a body is in equilibrium under the action of a system of forces, they may be all increased, or all diminished in any proportion, without affecting the equilibrium.

20. It scarcely needs remarking, that if a set of forces balance each other, any one of them is equal to, and acts in an opposite direction to, the resultant of all the rest.

21. It is proved by experiment, that when a rigid body is in equilibrium, any point (of the body) in the line of the direction in which a force acts, may be taken for the point of application of the force, without affecting the equilibrium.

22. If a system of unbalancing forces acts upon the same point of a rigid body, they will have the same resultant as if they acted upon a free particle.

CHAPTER I.

ON FORCES WHICH ACT IN ONE PLANE UPON A PARTICLE, OR UPON THE SAME POINT OF A RIGID BODY.

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23. *To find the resultant of several forces acting, in the same line, upon the same point of a rigid body.*

If all the forces act in the same direction along the line, they will produce the same effect as a single force equal to their sum.

If some act in one direction and some in the opposite direction, then by the first case the resultant of each set will be equal to the sum of the forces of which it is composed: and these two resultants, acting in opposite directions, will be equivalent to a single resultant equal to their difference. Hence then whether the original forces act in the same or in opposite directions, their resultant is equal to their algebraic sum.

In forming this sum, we are to account those forces positive which act in one direction, and those negative which act in the opposite direction; and the algebraic sign of the sum so formed will shew in what direction the resultant acts.

24. Cor. If a number of forces act in the same line upon the same point of a rigid body, they will be in equilibrium when their algebraic sum is equal to zero, for in that case their resultant vanishes, and they produce no effect. Hence the condition of equilibrium of any number of forces acting in the same line and upon the same point of a rigid body is *that their algebraic sum shall be equal to zero.*

25. Def. Lines are said to represent forces in magnitude and direction, when they are drawn parallel to the directions in which the forces act, and have their lengths proportional to the magnitudes of the forces.

26. *If from a point two lines be drawn representing two forces which act upon a point; and if upon these lines a parallelogram be constituted, the diagonal drawn from the same point will represent the resultant of the two forces.* This property is usually cited as "the parallelogram of forces."

We shall first prove that the diagonal represents the *direction* of the resultant force. This part of the proposition is evidently true when the two given forces are equal; let us assume that p, q and r are three forces, such that this is true for p and q; and also for p and r. At the point A, fig. 2, apply p in the direction AB: and q, r, both in the direction AD. Take AB, AC, CD to represent the respective magnitudes of these forces. Complete the parallelograms, and draw the diagonals as in the fig. The resultant of p and q acts in the direction AE, by hypothesis; and we may by Art. 21, suppose it to act at E; and we may there resolve it into its original components p and q; the latter acting in the line EF, and the former in the line CE produced; this we may suppose by Art. 21 to act at C, and the former at F; also the force r may be supposed to act at C. We have now two forces p, r at C represented by the lines CE, CD; their resultant by hypothesis acts in the line CF, and therefore we may suppose at F. The three forces p, q, r, which originally acted at A, are by this process reduced to forces acting at F. F is therefore in the line in which their resultant acts when they are applied at A, (Art. 21). Now AD, AB, represent two forces $q+r$ and p; and we have just shewn that their resultant acts in the direction of the diagonal AF. If then our proposition be true for the two forces p, q; and also for the two forces p, r; it is also true for the forces p and $q+r$. Now it is true when p, q, r are all equal; and hence it is true for p and $2p$: and because it is true for p, p; and also for p, $2p$; therefore it is true for p, $3p$;... and by following the same mode of reasoning it is true for p and mp, m being any whole number.

Again, because it is true for mp and p, and also for mp and p; therefore it is true for mp and $2p$; and as before for mp and np, n being an integer. Hence our proposition respecting the *direction* of the resultant is true for any two *commensurable* forces mp, np.

If the proposed forces P, Q be incommensurable, by taking p extremely small and the integers m, n correspondingly large, we can make mp differ from P, and np from Q by less than any quantities which can be assigned; and we may then use mp and np, instead of P and Q, without any sensible error; and therefore the proposition is true of P and Q.

We shall now prove that the diagonal represents the *magnitude* of the resultant force.

Let AC, AB (fig. 3) represent the magnitudes and directions of two forces acting on a point. Complete the parallelogram: its diagonal AE has been proved to represent the direction of the resultant; it also represents its magnitude. For in EA produced take AF to represent its magnitude; then AB, AC, AF represent three forces which balance each other: wherefore completing the parallelogram $AFGB$, its diagonal AG represents the direction of the resultant of AF, AB, and is consequently in the same line with AC (Art. 20). Hence $AGBE$ is a parallelogram, and therefore $AE = GB = AF$; that is, AE represents the magnitude of the resultant of AB, AC.

27. *If three forces acting on a point are represented by the sides of a triangle taken in order, they will balance each other.* And conversely; *If three lines, forming a triangle, be parallel to the directions of three forces which, acting on a point, balance each other, the sides of the triangle taken in order will represent the forces.*

For let AB, BE, EA (fig. 3) represent the forces P, Q, R which act on a point. Complete the parallelogram BC; then because AC is drawn parallel and equal to BE, it also represents the force Q. Hence the resultant of P and Q is represented by AE; which being equal, and in a contrary direction, to EA which represents R, there is equilibrium.

Conversely, let the three forces P, Q, R, acting on a point balance each other: and suppose ABE (fig. 4) to be the triangle whose sides are respectively parallel to the directions in which P, Q, R act. Two, at least, (AB, BE suppose) represent the *directions* of the corresponding forces; and we are at liberty to suppose that one of these (AB suppose) represents also the magnitude of its force P: if BE do not represent the magnitude of the other force Q, take BE' to represent it, and join AE'. Then (by the former case) P, Q and a force represented by $E'A$ will balance each other. But P, Q, R balance each other; and therefore R is represented by $E'A$ both in magnitude and direction; which is impossible (because EA represents R in direction by hypothesis) unless E' coincide with E. Therefore E' does coincide with E, and therefore the forces are represented by AB, BE, EA, which are the sides of the triangle taken in order.

28. *If three forces, acting upon a point, balance each other, their directions lie in a plane; and their magnitudes are respectively proportional to the sine of the angle between the directions in which the other two act.*

Let the forces be P, Q, R (fig. 5) acting in the directions OA, OB, OC. In OA, OB, take points A, B, such that OA, OB represent the magnitudes as well as the directions of P, Q. Complete the parallelogram $AOBD$, and join OD. Then Q being represented by OB may also be represented by AD; also as the three forces represented by OA, AD, DO, acting on a point will balance each other (Art. 27), therefore P, Q and a force represented by DO balance each other; but P, Q, R balance each other; therefore R is represented by DO: and consequently DOC is a straight line, and OA, OB, OC lie in the plane of the triangle OAD. Also

$$\begin{aligned} P : Q : R &:: OA : AD : DO \\ &:: \sin ODA : \sin DOA : \sin OAD \\ &:: \sin DOB : \sin AOC : \sin PAD \\ &:: \sin QOR : \sin POR : \sin POQ, \end{aligned}$$

or, $$\frac{P}{\sin QOR} = \frac{Q}{\sin POR} = \frac{R}{\sin POQ}.$$ Therefore, &c.

29. *Two forces act upon the same point in directions at right angles to each other, to find their resultant* (fig. 5*).

Let the forces be X, Y acting upon the point O in the directions Ox, Oy at right angles to each other. Take OM, ON to represent the forces; complete the rectangle $OMPN$, and draw the diagonal OP. This line by Art. 26 represents the resultant of X and Y. Let R be the resultant, and θ the angle POM which its direction makes with the direction of the force X.

$$\text{Now } OP^2 = OM^2 + MP^2;$$

$$\therefore\ R^2 = X^2 + Y^2,$$

which determines the value of R: and then the equation

$$\tan\theta = \frac{MP}{OM} = \frac{Y}{X}$$

determines the value of θ.

30. COR. If a force R be given and it be required to resolve it into two components acting in directions at right angles to each other, we must employ the equations

$$X = R\cos\theta, \text{ and } Y = R\sin\theta,$$

which are derived from the equations

$$OM = OP\cos\theta, \text{ and } MP = OP.\sin\theta.$$

31. *Any number of forces act upon a point in given directions in a plane; to find their resultant.*

Let F_1, F_2, $F_3 \ldots F_n$ be the forces, and O (see fig. of Art. 29) the point upon which they act. In the plane in which are the lines in which the forces act, draw any two lines Ox, Oy at right angles to each other; and denote by α_1, α_2, $\alpha_3 \ldots \alpha_n$ the angles which the directions of F_1, F_2, $F_3 \ldots F_n$ respectively make with Ox.

Then the components of these forces are respectively (by Art. 30),

$$F_1\cos\alpha_1,\ F_2\cos\alpha_2,\ F_3\cos\alpha_3 \ldots\ldots F_n\cos\alpha_n$$

in the direction Ox; and

$$F_1\sin\alpha_1,\ F_2\sin\alpha_2,\ F_3\sin\alpha_3 \ldots\ldots F_n\sin\alpha_3$$

in the direction Oy.

Let us replace (see Art. 15) the original forces

$$F_1,\ F_2,\ F_3 \ldots\ldots F_n$$

by these two sets of components. These components are respectively equivalent to two forces acting in the lines Ox, Oy (Art. 23), and being equal to

$$F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + F_3 \cos \alpha_3 + \dots + F_n \cos \alpha_n,$$

$$\text{and } F_1 \sin \alpha_1 + F_2 \sin \alpha_2 + F_3 \sin \alpha_3 + \dots + F_n \sin \alpha_n.$$

Let R be the result of the original forces

$$F_1,\ F_2,\ F_3 \dots F_n,$$

and suppose that θ is the angle which the line in which it acts makes with Ox. Then since R is equivalent to the original forces, it is also equivalent to the two components of them which have just been found: hence (by Art. 30)

$$R \cos \theta = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \dots + F_n \cos \alpha_n,$$

$$\text{and } R \sin \theta = F_1 \sin \alpha_1 + F_2 \sin \alpha_2 + \dots + F_n \sin \alpha_n.$$

From which two equations both R and θ may be found.

REMARK. The sum of a number of quantities of the same form is often, for brevity, represented by prefixing the symbol Σ to a term representing the general form. Upon this principle the above equations may be written thus:

$$R \cos \theta = \Sigma\,(F \cos \alpha), \text{ or } \Sigma\,.\,F \cos \alpha,$$

$$\text{and } R \sin \theta = \Sigma\,(F \sin \alpha), \text{ or } \Sigma\,.\,F \sin \alpha;$$

$$\therefore\ R^2 = (\Sigma\,.\,F \cos \alpha)^2 + (\Sigma\,.\,F \sin \alpha)^2,$$

$$\text{and } \tan \theta = \frac{\Sigma\,.\,F \sin \alpha}{\Sigma\,.\,F \cos \alpha}.$$

32. *To find the conditions of equilibrium of forces acting upon a point in any directions in one plane.*

Let F_1, $F_2 \dots F_n$ be the forces; α_1, $\alpha_2 \dots \alpha_n$ the angles which their directions make with a line Ox; then, proceeding as in the last article, we have the equations there found for the determination of their resultant. But because they balance each other by hypothesis, they have no resultant, and therefore $R = 0$, or

$$0 = (\Sigma\,.\,F \cos \alpha)^2 + (\Sigma\,.\,F \sin \alpha)^2.$$

But as the right-hand member consists of two terms which, being squares, are essentially positive, their sum cannot be equal to 0 unless each be separately equal to 0;

$$\therefore\ 0 = \Sigma . F \cos \alpha, \quad \text{and} \quad 0 = \Sigma . F \sin \alpha.$$

If the investigation in Art. 31 be examined, it will be seen that the line Ox was taken in any direction in the plane of the forces; and hence we may in the most general terms state the signification of the two equations just found to be as follows,

If any number of forces act in one plane upon a point, that there may be equilibrium,

The sums of the components of the forces parallel to any two lines, at right angles to each other, in the plane of the forces must be separately equal to zero.

The converse is evidently true also.

No other condition is necessary for equilibrium, for if $\Sigma . F \cos \alpha = 0$, and $\Sigma . F \sin \alpha = 0$, it follows inevitably that $R = 0$, and therefore there is equilibrium.

CHAPTER II.

ON FORCES WHICH ACT UPON A PARTICLE, OR UPON THE SAME POINT OF A RIGID BODY, IN ANY DIRECTIONS NOT IN ONE PLANE.

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33. *If three forces acting upon the same point be respectively represented by the three edges of a parallelopiped which meet, the diagonal of the parallelopiped drawn from that point to the opposite corner will represent their resultant.*

For let OA, OB, OC (fig. 6) be the edges which represent the three forces, and OE the diagonal of the parallelopiped: draw OD, CE.

Then because OA, OB represent two forces, OD represents a force which is equivalent to them both (Art. 26): hence the three forces represented by OA, OB, OC are equivalent to the two represented by OD, OC, which again are equivalent to the single force represented by OE, for CD is a parallelogram.

34. *Three forces act upon a point in directions which are at right angles to each other; to find their resultant.*

Let X, Y, Z be the forces, acting upon the point O (fig. 7) in the lines Ox, Oy, Oz which make right angles yOz, xOy, zOx with each other. From O set off OL, OM, ON to represent the forces X, Y, Z respectively. Complete the parallelograms $OMQL$, $OQPN$, and join OP; this line, by the last Art., represents the resultant required.

Let R denote the resultant, and α, β, γ the angles POx, POy, POz which its direction makes with the directions of the given forces.

Then because $ON = OP \cdot \cos\gamma$;

$$\left.\begin{aligned} \therefore\ Z &= R\cos\gamma \\ \text{Similarly}\quad Y &= R\cos\beta \\ \text{and}\quad X &= R\cos\alpha \end{aligned}\right\} \ldots (\text{A});$$

$$\begin{aligned} \therefore\ X^2 + Y^2 + Z^2 &= R^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) \\ &= R^2\left(\frac{OL^2}{OP^2} + \frac{OM^2}{OP^2} + \frac{ON^2}{OP^2}\right) \\ &= R^2\,\frac{OL^2 + LQ^2 + QP^2}{OP^2} = R^2\,\frac{OQ^2 + QP^2}{OP^2} \\ &= R^2\,\frac{OP^2}{OP^2} = R^2. \end{aligned}$$

This equation gives the value of R, and then the three equations marked (A) give the angles α, β, γ, which fix the line in which R acts.

Remark. The reader will observe from the above that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

a formula which will be of frequent use in the following pages.

35. Cor. If a force R be given, and it be required to resolve it into three components, whose directions are at right angles to each other, we must employ the equations marked (A).

36. *Any number of forces act in given directions upon a point; to find their resultant.*

Let O (fig. 7) be the point upon which the given forces F_1, F_2, F_3 ... F_n act; from O draw three lines Ox, Oy, Oz, arbitrarily taken, at right angles to each other; and denote by $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$ ... $\alpha_n\beta_n\gamma_n$, the angles which the directions of the forces make with these three fixed lines.

The respective components of the given forces are

$$F_1\cos\alpha_1,\ F_2\cos\alpha_2,\ \ldots\ F_n\cos\alpha_n,$$

in the direction Ox;

$$F_1 \cos\beta_1, \quad F_2 \cos\beta_2, \dots F_n \cos\beta_n,$$

in the direction Oy; and

$$F_1 \cos\gamma_1, \quad F_2 \cos\gamma_2, \dots F_n \cos\gamma_n,$$

in the direction Oz.

Replace now the original forces by these three sets of components (Art. 15); each set is reduced to one force by Art. 23; and we then have three component forces

$$\Sigma \,.\, F\cos\alpha, \quad \Sigma \,.\, F\cos\beta, \quad \Sigma \,.\, F\cos\gamma,$$

acting respectively in the lines Ox, Oy, Oz.

Let R be the resultant required, and α', β', γ' the angles which the line in which it acts makes with Ox, Oy, Oz. Then since R is equivalent to the original forces, it is also equivalent to the three components of them which have just been found; hence

$$\left.\begin{aligned} R\cos\alpha' &= \Sigma \,.\, F\cos\alpha \\ R\cos\beta' &= \Sigma \,.\, F\cos\beta \\ R\cos\gamma' &= \Sigma \,.\, F\cos\gamma \end{aligned}\right\} \dots \text{(A)};$$

and therefore since $1 = \cos^2\alpha' + \cos^2\beta' + \cos^2\gamma'$, (Art. 34, Rem.) we find

$$R^2 = (\Sigma \,.\, F\cos\alpha)^2 + (\Sigma \,.\, F\cos\beta)^2 + (\Sigma \,.\, F\cos\gamma)^2.$$

This equation gives the value of R; and then the equations (A) will give α', β', γ', which fix the direction in which R acts.

37. *To find the equations of the line in which the resultant acts.*

Suppose OP (fig. 7) the line in which the resultant acts; and let x, y, z be the co-ordinates of any point P in it.

Then if OP be taken to represent R, the co-ordinates will represent the components, and therefore by Art. 25,

$$\frac{x}{\Sigma \,.\, F\cos\alpha} = \frac{y}{\Sigma \,.\, F\cos\beta} = \frac{z}{\Sigma \,.\, F\cos\gamma},$$

which are the equations required.

If the point at which the forces act be not the origin of co-ordinates, let its co-ordinates be a, b, c; then since the line whose equations are required passes through this point,

$$\frac{x-a}{\Sigma . F \cos \alpha} = \frac{y-b}{\Sigma . F \cos \beta} = \frac{z-c}{\Sigma . F \cos \gamma}$$

are the equations of it in this case.

38. *To find the conditions that a system of forces acting upon a point in any directions, may balance each other.*

It is evident that there cannot be equilibrium among the forces $F_1, F_2, F_3 \ldots\ldots F_n$, unless their resultant be evanescent, and therefore we must have

$$0 = (\Sigma . F \cos \alpha)^2 + (\Sigma . F \cos \beta)^2 + (\Sigma . F \cos \gamma)^2,$$

which for a reason similar to that assigned in Art. 32 resolves itself into the three independent conditions

$$0 = \Sigma . F \cos \alpha, \quad 0 = \Sigma . F \cos \beta, \quad 0 = \Sigma . F \cos \gamma.$$

Or, in words, (remembering that the positions of Ox, Oy, Oz were arbitrarily chosen),

The sum of the components of the given forces parallel respectively to any three lines at right angles to each other must separately be equal to zero.

CHAPTER III.

ON FORCES WHICH ACT IN ONE PLANE BUT NOT UPON THE SAME POINT OF A RIGID BODY.

THE THEORY OF COUPLES.

39. Remark. It has been stated in Art. 21, that the effect of a force is not altered by supposing it to be transferred from one point of the body in the line of the direction of its action to another: from this it follows that if the directions of the forces which act at different points of a rigid body, all pass through a point, we may fictitiously transfer them to that point, and then by the preceding Chapters find their resultant, which in its turn we may transfer to any convenient point of the rigid body which happens to lie in the line of its direction. It is obvious, that when any two forces in the same plane act upon a rigid body at different points, their directions unless parallel being produced will meet, and therefore after the statement just made it will not be necessary to include the consideration of two non-parallel forces in the present Chapter, we shall therefore begin with the following.

40. *Two forces act in parallel directions upon different points of a rigid body, to find their resultant.*

Case 1. Let F, F' be the two forces, and let us, first, suppose them to act in the same direction.

Let A, B (fig. 8) be any two points of the rigid body in the lines of direction of the respective forces: join A, B; at these points in opposite directions along the line AB apply any two equal forces f, f'. These being in equilibrium produce no effect.

Now F and f (by Art. 26) and F' and f' will have resultants (m, n suppose) acting in certain directions Am, Bn within the angles FAf, and $F'Bf'$: these lines being produced will meet in some point P to which let m, n be transferred: and let them there be resolved into their original components; viz. m into f and F, acting at P in the directions Pf and PR (PR being drawn parallel to AF); and n into f' and F' acting at P in the directions Pf' and PR, which is also parallel to BF'. The forces f and f' at P being in equilibrium may be removed, and there remain the original forces F, F' both acting at P along the line PR parallel to their direction at A and B. Hence the resultant of F and F' is a force, equal to their sum $F+F'$, acting at any point in the line PR; the position of which we find as follows.

Let PR cut AB in Q. Then because m is the resultant of F and f, a force equal to m applied at A in the direction AP would keep the two forces F, f in equilibrium; and the three being parallel to the sides of the triangle APQ taken in order, are proportional to those sides (Art. 27);

$$\therefore\ F : f :: PQ : AQ.$$

Similarly $$f' : F' :: BQ : PQ$$

$$\therefore\ F : F' :: BQ : AQ; \quad \because f=f'.$$

Consequently Q divides AB into two parts which are inversely proportional to the forces adjacent to which they lie.

41. CASE 2. Let us now suppose the two forces F, F' to act in contrary directions, as in fig. 9, and that they are unequal, F being the greater.

Introduce the equal and opposite forces f and f' as before; and let m be the resultant of F and f; and n that of F' and f'. Then since the angle FAf is equal to the angle $F'Bf'$, it will be found, by constructing the parallelograms of force upon FA, Af, and $F'B$, Bf', that since F is greater than F', the direction of the resultant m lies nearer to AF, than the direction of the resultant n to BF': that is, the angle fAm is greater than the angle $f'Bn$ or ABP. Consequently the lines nB, Am being produced will meet on the side towards the greater force F, as is represented in the figure. From this point proceeding as in the former case we find that the forces F, F' preserving their proper directions, may be removed to the point P. Hence their resultant R is equal to $F-F'$, the algebraic *sum* of the forces, and acts in the direction of the greater force. The word *sum* is used in the statement of this result, because F being assumed positive, F' acting in the contrary direction must be accounted a negative force. (See Art. 23.)

The position of the point Q is found as before from the proportion

$$F : F' :: BQ : AQ,$$

and it is to be noticed particularly that Q lies in BA *produced*; and is situated nearer to A (the point of application of the greater force) than to B.

42. Case 3. *Let us lastly suppose the two forces* F, F′ *acting in contrary directions, to be equal* (fig. 9).

In this case the angles fAm, $f'Bn$ are equal; and consequently the angles fAm, ABP are equal, and the lines Am, nB are parallel, and have no point of concourse. It would appear then, that the former mode of finding the resultant of F and F' fails entirely in this case. The present case may, however, be considered the ultimate state of Case 2, at which we arrive by supposing the magnitude of F' to approach continually nearer to that of F, until at length their difference becomes less than any assignable quantity. Let us then reconsider Case 2. We have found

$$\left\{\begin{aligned} &R = F - F', \\ &\text{and} \quad F : F' :: BQ : AQ; \\ &\therefore\ BQ = \frac{F \,.\, AB}{F - F'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)} \\ &\text{or}\ R \,.\, BQ = F \,.\, AB \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2).} \end{aligned}\right.$$

Hence, we see from (1) that as F' increases, the point Q moves continually farther from B, and BQ becomes infinite in the ultimate state; and at the same time from (2) we see that the resultant R diminishes in such a manner that the product

$R.BQ$ never changes; and R becomes zero in the limit. Hence, in the ultimate state, that is, when F' differs from F by less than any assignable quantity, we have a resultant zero acting at a point infinitely distant from A or B; yet even then the product $R.BQ$ remains finite, which apart from any other consideration would induce us to conjecture, that some finite effect is due to the action of F and F' in this case, although not such an one as can be represented by a single force.

Upon these grounds we conclude, *that a system of two equal forces acting in contrary directions on different points of a rigid body is not reducible to a single resultant.*

43. DEFS. A system of two equal forces acting upon a rigid body in opposite directions but not in the same straight line, is denominated A COUPLE. A plane which passes through the two lines in which the forces of a couple act, is called *the plane of the couple.*

When the line AB (fig. 9) is drawn at *right angles* to the directions of the forces of the couple, it is called *the arm of the couple;* and the product $F.AB$ (see (2) of Art. 42) is then called *the moment of the couple.*

44. REMARK. It is obvious from an examination of fig. 9, that one effect of a pair of forces, acting in contrary directions at different points of a rigid body, whether they be equal or unequal, is the communication of a rotatory motion (see Art. 6) to the body on which they act; what other effect they would produce is not so obvious, nor indeed does it belong to us, in treating of the present subject, to consider what is the effect of unbalanced forces in any case. For the satisfaction of the reader, however, and for convenience in what follows, it may be stated, that it is proved in *Dynamics*, that the sole effect of a couple is to communicate to the body on which it acts an angular motion about an axis passing through a certain point in the body, called the centre of gravity.

45. DEF. If the forces of the couple act so as to tend to turn the body round in the direction of the motion of the hands of a watch, it is called a right-handed couple, and more frequently a *positive* couple; but if, as in fig. 9, the forces act so as

to turn the body in the contrary direction, the couple is styled left-handed, or *negative*.

These terms, to prevent confusion, will be used in this book as here defined; but the reader will observe that, in Statics as in Algebra, the terms positive and negative are only relative, and may be applied, at discretion, to any two forces acting in contrary directions, or to any two couples which tend to communicate opposite angular motions to the body on which they act.

46. The reflecting reader will have remarked that a couple, though positive when viewed by a spectator looking at it from one position, appears negative to a spectator looking at it from a position on the other side of its plane. A couple is therefore positive or negative, according to the situation of the spectator, with respect to its plane. It will prevent confusion, if we call that face of the couple's plane the *positive* face, upon which the spectator looks when the couple appears to him to be positive: the other face of the plane must then be considered negative.

47. DEFS. A straight line, in length proportional to the moment of a couple, being drawn perpendicular to the plane of the couple, is called "*the axis of the couple.*"

And it is said to be the *positive*, or the *negative axis*, according as the perpendicular stands on the positive, or on the negative face of the couple's plane.

If the axis of a couple is mentioned without its being stated whether it is positive or negative, we are to understand that the positive axis is alluded to.

The angle between the planes of two couples is measured by the angle between their positive axes.

48. *The effect of a couple acting on a rigid body is not altered by turning its arm through any angle in the plane of the couple.*

Let F and F' be the equal forces of a couple acting at two points in the lines FA, $F'B$ (fig. 10), and having the arm AB. From A, any point in the line in which F acts, draw in an arbitrary direction in the plane of the couple AB' equal to AB;

and at A, B' in the plane of the couple F, F', and in directions at right angles to AB' apply two pairs of opposite forces f, g; f', g': each force being equal to F or F'. These being in equilibrium, produce no effect.

$B'g'$ and $F'B$ will intersect each other in some point C; join AC. Then because $AB = AB'$, and AC is common to the triangles ABC, $AB'C$, and the $\angle B = \angle B'$; therefore AC bisects the angles BCB', BAB': hence the resultant of the two equal forces F' g' which we may suppose to act at C lies in AC produced; and that of the equal forces F, g lies in CA produced. But, because AF is parallel to CF' and Ag parallel to Cg', therefore the $\angle FAg = \angle F'Cg'$; and consequently the resultant of the forces F', g' is equal to that of the forces F, g: we have just proved also that they act in opposite directions; therefore, the four forces F, g, F', g' balance each other, and may be removed.

There is then left only the couple f, f', which is the same as if the arm of the original couple had been turned through the arbitrary angle BAB'.

49. *The effect of a couple acting on a rigid body is not altered by removing it to any other part of the rigid body, provided its new plane be parallel to its original plane* (fig. 11).

Let F, F' be a couple acting upon a rigid body in the plane HH; let AB be its arm. Let KK be any other plane, in the rigid body, parallel to HH; and in this plane draw the line ab, parallel and equal to AB. At a and b apply pairs of opposite forces f, g; f', g': each force being equal and parallel to the forces F, F'. These pairs of forces balance each other, and therefore produce no effect on the rigid body. Draw Ab, Ba; they, being in the plane which contains AB and ab, necessarily intersect in some point C. In fact, if A, a, and B, b were joined, $AabB$ would be a parallelogram, and therefore Ab, Ba, being its diagonals, bisect each other in C. Draw PCQ parallel to AF. Then because $F = g'$, and $bC = AC$,

$$\therefore F : g' :: bC : AC.$$

Hence F and g' (by Art. 40) have a resultant $F + g'$, which acts at C in the line CP. Similarly it may be shewn, that F'

and g have a resultant, $F'+g$ acting at C in the line CQ. Now $F+g'=F'+g$ and CP is opposite to CQ, therefore the four forces F, g', F', g balance each other, and may be removed. There remains then only the couple f, f', which is the same as if the original couple had been removed into the new plane KK, retaining its arm ab parallel to AB; but we may now (by Art. 48) turn the arm ab through any angle without altering the effect of the couple. And hence the effect of a couple, &c.

50. *The effect of a couple acting on a rigid body is not altered by removing it to any other part of the rigid body in its own plane.*

The demonstration in the last Article will serve for this, using fig. 12 instead of fig. 11.

51. *A couple acting on a rigid body may be changed for any other couple acting upon the same rigid body, provided the moments of the two couples be equal, their planes parallel, and they be both of the same kind, i. e. both positive or both negative.*

Let HH (fig. 11) be the plane of the couple F, F'; and in any other plane KK of the rigid body draw, parallel to AB, a line ab of any proposed length: at a, b apply pairs of equal and opposite forces f, g; f', g'; of such magnitude that

$$F\,.\,AB=f\,.\,ab,$$

these balance each other, and therefore produce no effect.

Now AB and ab being parallel, the lines Ab, Ba lie in one plane, and intersect in some point C: and because AB is parallel to ab, the $\angle\, CAB=\angle\, Cba$, and the $\angle\, CBA=\angle\, Cab$; consequently the two triangles ACB, bca are similar. Now because $f=g'$

$$\therefore\ F : g' :: ab : AB$$
$$:: Cb : CA;$$

therefore, by Art. 40, the resultant $F+g'$ of the two forces F, g' acts at C in the direction CP. In a similar way it may be shewn that $F'+g$, the resultant of F', g, acts at C in the direction CQ. Now $F=F'$ and $g'=g$, and therefore $F+g'=F'+g$; consequently the four forces F, g', F', g are in equilibrium, and may be removed; which being done, the original couple is replaced by the equivalent couple f, f' whose arm is ab. This couple f, f' may now be turned through any angle in the plane KK, and thus the proposition is established.

52. *Any number of couples act upon a rigid body in the same plane, or in parallel planes; to find their resultant.*

Change all the couples into others equivalent to them, and therefore of the same moment, but all having their arms of the length b. Then if $F_1, F_2, F_3 \ldots F_n$ be the forces; and $a_1, a_2, a_3 \ldots a_n$ the arms of the original couples; and $P_1, P_2, P_3, \ldots P_n$ the forces of the corresponding equivalent couples, we shall have

$$P_1 b = F_1 a_1, \quad P_2 b = F_2 a_2, \ldots P_n b = F_n a_n.$$

Now since the new couples act in parallel planes and have equal arms, they may be removed into the same plane, and then turned round and transposed so as to make all their equal arms exactly coincide; in which position the system of couples is reduced to one couple, the arm of which is b, and the forces of which are equal to

$$P_1 + P_2 + P_3 + \ldots + P_n.$$

Hence the moment of the resultant couple

$$= (P_1 + P_2 + P_3 + \ldots + P_n) \,.\, b$$

$$= F_1 a_1 + F_2 a_2 + F_3 a_3 + \ldots + F_n a_n$$

$=$ the sum of the moments of the original couples.

Whence, *the moment of the resultant couple is equal to the sum of the moments of the original couples.*

The reader will be careful to remark, that if any of the couples are of a negative character, their moments are to be accounted negative in taking this sum.

53. Cor. If all the n couples be equal, the moment of their resultant couple is n times the moment of one of them;

and as the effect of n equal couples must be n times the effect of one of them, it follows that the moment of a couple is a proper measure of its effect in producing or destroying equilibrium. Whenever, therefore, we have occasion to speak of the *magnitude* of a couple, we shall do so by stating its *moment;* thus, the couple G will signify the couple whose moment is G. It will lead to no inconvenience that the magnitude of the forces which compose the couple is not stated, seeing that the effects of all couples of equal moments acting in the same plane, whatever be the magnitudes and directions of their forces, are the same. It will also be observed, that it is not necessary to state the precise plane in which a couple is situated; it will be sufficient to know its moment, and the position of some line to which its axis is parallel.

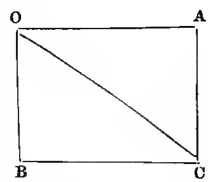
54. It will be observed, that all equivalent couples have their axes equal and parallel.

55. *If from a point two straight lines be drawn parallel and equal to the axes of two couples, and upon them a parallelogram be described, the diagonal drawn from the same point will be parallel and equal to the axis of the resultant couple.* (This proposition is usually cited as the parallelogram of couples.) (fig. 13).

As the planes (HOA, HOB, suppose) of the couples are not parallel, let them intersect in the line HO. Change the couples into two equivalent couples having their forces FF', ff' all equal; place these new couples so that one extremity of their arms OA, OB shall be at O, and the forces F, f which act there, shall act in the line OH, as in the figure. Complete the parallelogram $OADB$, and draw the diagonals OD, AB, bisecting each other in C. Then because F' and f' are equal and act in the same direction, they are equivalent to a resultant $F'+f'$ acting at C (Art. 40). But such a force at C would likewise be the resultant of the same forces F', f' acting at D, O. We may therefore transpose F' to D, and f' to O, which being done, f' and f at O balancing may be removed; and there will only remain F at O and F' at D, forming a couple F, F' whose arm is OD, which is therefore the resultant of the two original couples.

Now the forces of the two component couples and of their resultant being equal, their axes which are proportional to their moments, are in this case proportional to their arms OA, OB, OD; we may therefore consider OA, OB, OD as being equal to the axes. If therefore from O in the plane OBA, we draw three lines respectively perpendicular and equal to OA, OB, OD, they will be the axes of the three couples, and will then have the same position as the lines OA, OB, OD would take if the parallelogram $OADB$ were turned through a right angle about the fixed line OH. This figure $OADB$ so turned is the parallelogram stated in the enunciation of the proposition to possess the property which we have just proved belongs to it.

56. *Two couples act upon a rigid body in planes which are at right angles to each other; to find their resultant.*



From any point O, draw OA, OB equal and parallel to the axes of the two couples. Complete the rectangle $OACB$, and draw its diagonal OC. By the last article OC is equal and parallel to the axis of the resultant couple.

Let L, M, G be the moments of the two component couples and of their resultant. $\theta = COA$ the angle at which the axis of G is inclined to that of L.

Then because $OA = OC \cos\theta$, and $OB = OC \sin\theta$;

$$\therefore\ L = G\cos\theta, \quad \text{and } M = G\sin\theta,$$

from which we find

$$G^2 = L^2 + M^2,$$

$$\text{and } \tan\theta = \frac{M}{L};$$

which equations determine both the magnitude of the resultant couple, and the position of its axis.

57. If it should be required to resolve a given couple whose moment is G into two components acting in planes at right angles to each other, we must use the equations

$$L = G\cos\theta, \quad M = G\sin\theta.$$

58. *Any number of forces act on a rigid body in parallel directions in one plane at different points of the body; to find their resultant.*

Let F_1, $F_2 \dots F_n$ be the forces; from any point O (fig. 14) of the rigid body in the plane of the forces draw a line cutting their directions perpendicularly in the points A, $B \dots H$; and put

$$OA = x_1, \quad OB = x_2, \dots \; OH = x_n.$$

At O apply two opposite forces each equal and parallel to F_1; they do not affect the system. In the same way apply at O a pair of forces for each of the remaining forces F_2, $F_3 \dots F_n$. By this means we have n forces acting at O in the direction OR, respectively equal to F_1, F_2, ... F_n; these are equivalent to a single force R acting at O in the direction OR, and

$$R = F_1 + F_2 + \dots + F_n = \Sigma . F.$$

We have, besides, n couples whose arms are x_1, $x_2 \dots x_n$, which are (by Art. 52) equivalent to a single couple, whose moment

$$= F_1 x_1 + F_2 x_2 + \dots + F_n x_n$$
$$= \Sigma . Fx.$$

Consequently the given system of forces is equivalent to a single force $\Sigma . F$ acting at O in the direction OR; and a negative couple whose moment is $\Sigma . Fx$.

59. The result just obtained is perfectly general, but admits of simplification except in the particular case when $\Sigma . F = 0$.

(1) In the particular case when $\Sigma . F = 0$, there is no resultant force acting at O, and therefore the only resultant is the couple whose moment is $\Sigma . Fx$.

(2) When $\Sigma . F$ is not $= 0$, change the couple whose moment $= \Sigma . Fx$ into an equivalent couple which has its forces R', R'' equal to R or $\Sigma . F$, and place it so that its arm OK (fig. 14) shall coincide with the line OH;

$$\therefore \; (\Sigma . F) . OK = \Sigma . Fx \; \dots\dots \; \text{(Art. 50.)}$$

By the arrangement the force R at O is balanced by R'' one of the forces of our new couple; and these being removed, there remains only the force $R' = \Sigma . F$ at the point K determined by the equation

$$OK = \frac{\Sigma . Fx}{\Sigma . F}.$$

Consequently, except when $\Sigma . F = 0$, the resultant is a single force equal to $\Sigma . F$ acting at the point just found.

60. Cor. If the line OH instead of cutting the directions of the forces at right angles, should cut them in an $\angle\ \alpha$, we should have found that

(1) When $\Sigma . F = 0$, the resultant is a couple whose moment is $(\Sigma . Fx) \sin \alpha$: and

(2) When $\Sigma . F$ is not $= 0$, the resultant is a single force $\Sigma . F$ acting at the point determined by the same equation as before, viz.

$$OK = \frac{\Sigma . Fx}{\Sigma . F}.$$

61. *Any number of parallel forces act upon a rigid body in one plane at different points of the body; to find the conditions that they may balance each other.*

Let the system of forces be that of Art. 58; then we have to consider the two cases pointed out in the last Article. In the second case the resultant is the force $\Sigma . F$ acting at K; and there cannot be equilibrium unless this force vanish, or $\Sigma . F = 0$. But if this be the case, the second case coincides with the first; and the resultant is a couple whose moment $= \Sigma . Fx$: there cannot be equilibrium therefore unless this couple also vanish. Consequently the conditions of equilibrium are

$$\Sigma . F = 0, \text{ and } \Sigma . Fx = 0;$$

these are both necessary and sufficient for equilibrium. They are *necessary*, for if the former only be satisfied, there will exist the couple of Case 1; and if the latter only be satisfied, there will exist the resultant force acting at O. And they are *sufficient*, for they secure that there shall exist neither the resultant of Case 1, nor that of Case 2.

62. Def. The products F_1x_1, F_2x_2, ... F_nx_n are called the *moments* of the forces F_1, F_2, ... F_n about the point O; they are also called the moments of the same forces about an axis passing through O at right angles to the plane of the forces.

Hence remembering that the point O was arbitrarily chosen in the plane of the forces, the two conditions of equilibrium of parallel forces acting on a rigid body in one plane may be thus enunciated in words:—

The algebraic sum of the forces, and the sum of the moments of the forces about any point in the plane of the forces or about any axis perpendicular to the plane of the forces, must be each equal to zero.

63. *Suppose that there is in the plane of the forces a fixed point, or in the body a fixed axis not parallel to the plane of the forces: to find the conditions of equilibrium.*

If there be a fixed point in the plane of the forces, let that point be taken for O; or if there be a fixed axis it will *cut* the plane of the forces in a point, which take for O; then the investigations of Art. 58 apply here. The force $\Sigma . F$ which acts at O, can produce no effect since it acts on an immoveable point; it is not necessary therefore that $\Sigma . F$ should be $= 0$. But the couple whose moment $= \Sigma . Fx$, if it exist, will turn the body about O; and therefore that there may be equilibrium, it is necessary and sufficient that

$$\Sigma . Fx = 0;$$

hence there is only one condition of equilibrium in this case, which we may thus express in words:—

The sum of the moments of the forces about the fixed point, or about that point where the fixed axis cuts the plane of the forces, must be equal to zero.

Remark. When there is in the plane of the forces a fixed point, and the forces are in equilibrium, the pressure on the fixed point $= \Sigma F$, which is the same as if every force were transposed to that point, without altering the direction in which it acts.

64. *Any number of forces act upon a rigid body in one plane, at different points of the body and in directions not necessarily parallel; to find their resultant.*

Let F_1, F_2, ... F_n be the forces, and in the plane in which they act, from a point O, arbitrarily chosen, draw any two lines Ox, Oy at right angles to each other. To these lines as co-ordinate axes refer the given forces and their points of application (fig 15).

Let α_1, α_2 ... α_n be the inclinations of the lines in which the forces act to Ox; x_1y_1, x_2y_2, ... x_ny_n the co-ordinates of the points of application of the forces; if P be that of F_1, then $x_1 = OM$, $y_1 = PM$. From the point O draw OQ perpendicular to F_1P; and at O apply two opposite forces F', F'' each equal and parallel to F_1. By this means we have a force F' acting at O, and a couple $(F_1F''_,)$ whose moment is equal to $-F_1 . OQ$. Or we may say that the force F_1 may be transposed to O without altering its direction, if at the same time we also apply to the body a couple whose moment

$$= - F_1 . OQ$$

$$= - F_1 . (ON - QN), \text{ since } MN \text{ is parallel to } PQ,$$

$$= - F_1 . (x_1 \cos MON - y_1 \sin MPQ)$$

$$= - F_1 . (x_1 \sin \alpha_1 - y_1 \cos \alpha_1)$$

$$= (F_1 \cos \alpha_1) . y_1 - (F_1 \sin \alpha_1) . x_1;$$

$$\text{or } = X_1y_1 - Y_1x_1,$$

if we put X_1 Y_1 for the components of F_1 parallel to the co-ordinate axes Ox, Oy.

The same method being applied in succession to each one of the remaining forces of the system, we shall have transposed all the forces to O, each preserving its original direction; but there will be acting on the body besides them a number of couples in one plane whose moments are

$$X_1y_1 - Y_1x_1, \;\; X_2y_2 - Y_2x_2 \ldots X_ny_n - Y_nx_n.$$

If G be the resultant of the couples, and R the resultant of the forces at O, we shall have

$$G = \Sigma\,(Xy - Yx) \;\ldots\; \text{Art. 52},$$

$$\text{and } R^2 = (\Sigma\,.\,F\cos\alpha)^2 + (\Sigma\,.\,F\sin\alpha)^2 \ldots \text{Art. 31},$$

$$= (\Sigma\,.\,X)^2 + (\Sigma\,.\,Y)^2$$

$$\text{and } \tan\theta = \frac{\Sigma\,.\,Y}{\Sigma\,.\,X},$$

θ being the inclination of the line in which R acts to Ox.

65. The result just obtained is perfectly general, but it can be simplified, being reducible to a single resultant, except when $R = 0$, i.e. except when $\Sigma X = 0$ and $\Sigma Y = 0$.

For, (1) When $\Sigma X = 0$ and $\Sigma Y = 0$, there is no resultant force acting at O, and the only resultant is the couple whose moment $= (Xy - Yx)$.

(2) When the two equations $\Sigma X = 0$, and $\Sigma Y = 0$ are not both satisfied, change the couple whose moment is $\Sigma\,(Xy - Yx)$ into an equivalent couple which has each of its forces R' R'' equal to R, and place it so that one end of its arm OK (fig. 16) shall be at O, and one of its forces (R'') exactly opposite to R. R and R'' balance each other and may be removed; and there remains only the force R' acting at the point T such that

$$OT\,.\cos\theta = OK;$$

$$\therefore\; R\,.\,OT\,.\cos\theta = R\,.\,OK,$$

or, since $R\cos\theta = \Sigma X$ (Art. 31), and $R\,.\,OK = \Sigma\,(Xy - Yx)$,

$$OT = \frac{\Sigma\,(Xy - Yx)}{\Sigma X}.$$

Consequently when the two equations $\Sigma X = 0$, $\Sigma Y = 0$ are not both satisfied, the resultant is a single force R acting at the point just found, or at any point in the line KR'.

REMARK. From this it appears that when non-parallel forces, acting in one plane on a rigid body, admit of a single resultant, there is a certain line to any point of which all the forces admit of being transposed (each force retaining its original direction) without their effect being in any respect altered. This line is $R'K$, and we shall shew in the following Article how its equation may be found.

66. *When the forces of Art. 64 are reducible to a single resultant force, to find the equation of the line in which it acts.*

Let x' y' be the co-ordinates of any point in the line $R'K$ (fig. 16) in which the resultant acts. Then because this line passes through the point T and, being parallel to OR, makes an angle θ with the axis Ox, its equation is

$$y' - OT = \tan\theta \,.\, x',$$

$$\text{or } y' - \frac{\Sigma_{\prime}(Xy - Yx)}{\Sigma_{\prime} X} = \frac{\Sigma_{\prime} Y}{\Sigma_{\prime} X} \,.\, x',$$

$$\text{or } y'\Sigma_{\prime} X - x'\Sigma_{\prime} Y = \Sigma_{\prime}(Xy - Yx).$$

67. *Any number of forces act on a rigid body in one plane at different points of the body; to find the conditions that they may balance each other.*

Let the system of forces be that of Art. 64, then we have to consider the two cases of Art. 65. In the second of these cases the resultant is the force R' ($=R$) acting at T; there cannot be equilibrium unless this force vanish, or $R = 0$. But if this be the case, the second case coincides with the first; and the resultant is a couple whose moment $= \Sigma_{\prime}(Xy - Yx)$; there cannot be equilibrium unless this couple also vanish. Consequently the conditions of equilibrium are

$$\Sigma_{\prime} X = 0, \quad \Sigma_{\prime} Y = 0, \quad \Sigma_{\prime}(Xy - Yx) = 0.$$

These three conditions are both necessary and sufficient.

By referring to Art. 64 we perceive that $X_{\prime}y_{\prime} - Y_{\prime}x_{\prime}$ is equal to the moment of $F_{\prime}$ about the point O, consequently $\Sigma_{\prime}(Xy - Yx)$ is equal to the sum of the moments of all the forces about O. If then we remember that the point O, and the directions of the axes Ox, Oy, were arbitrarily chosen in the plane of the forces, we may enunciate the conditions of equilibrium as follows:

The algebraic sums of the components of the forces parallel to any two lines at right angles to each other in the plane of the forces must be each equal to zero; and the sum of the moments of all the forces about any point in the plane of the forces, or about any axis at right angles to the plane of the forces, must also be equal to zero.

68. *Suppose that there is in the plane of the forces a fixed point, or perpendicular to the plane of the forces a fixed axis; to find the conditions of equilibrium.*

Let the fixed point, or the point where the fixed axis cuts the plane of the forces, be taken for the point O in the investigation of Art. 64. Then the force R which acts at O can produce no effect since it acts on an immoveable point, it is not necessary then that R should be equal to zero. But the couple whose moment is $\Sigma\,(Xy - Yx)$, if it exist, will turn the body about O, and therefore that there may be equilibrium it is necessary and sufficient that

$$\Sigma\,(Xy - Yx) = 0.$$

There is therefore in this case only one necessary condition of equilibrium, viz;—

That the sum of the moments of all the forces about the fixed point or axis should be equal to zero.

Remark. When there is equilibrium, that is, when the above condition is satisfied, the pressure on the fixed point is due entirely to the force R which acts directly upon it. Hence the pressure on the fixed point is the same as if all the forces which act on the body were transposed to the fixed point without altering their directions.

CHAPTER IV.

ON FORCES, NOT IN ONE PLANE, WHICH ACT UPON DIFFERENT POINTS OF A RIGID BODY.

69. If the directions of the forces all pass through a point we may transfer them to that point, and find their resultant by Chapter I. or II.

70. In the present Chapter we shall meet with couples of which the planes are not parallel. We can however always reduce them to other couples in the planes of rectangular co-ordinates. It is necessary therefore only to observe that when a couple acts in a co-ordinate plane, it will be considered a positive couple when its axis stands on the positive side of that plane. Thus a positive couple

in the plane yz has its axis coinciding with $+ Ox$,

.... xz .... $+ Oy$,

.... xy .... $+ Oz$.

71. *Parallel forces not in one plane act on different points of a rigid body; to find their resultant.*

Take any point O (fig. 17) in the rigid body, from which draw Oz parallel to the direction of the proposed forces, which take for axis of z. Draw Ox, Oy in any directions at right angles to each other and to Oz, and take them for the axes of x and y. Let $Z_1, Z_2 \dots Z_n$ be the forces; P the point where the line in which Z_1 acts cuts the plane xy. $OM = x_1$, $MP = y_1$ the co-ordinates of P. Complete the parallelogram $OMPN$ and join OP. At the point O apply two pairs of opposite forces Z', Z'' each equal and parallel to Z_1; these do not affect the system.

Now it has been shewn in Art. 55, that when two equal parallel forces act in the same direction at the extremities of one of the diagonals of a parallelogram, they may be transposed to the extremities of the other diagonal. Let us on this principle transpose Z_1 to M, and Z' to N. We have then, *one* force Z' acting at O in the direction Oz, and two couples in the plane xz, yz whose arms are OM, ON, the former couple being negative. By this means we have transposed the force Z_1 to O, retaining its proper direction, and have introduced the couples $-Z_1x_1$, $+Z_1y_1$ in the planes xz, yz respectively. Proceeding in the same manner with the remaining forces Z_2, $Z_3 \ldots Z_n$, we shall have, instead of the original system, the forces Z_1, $Z_2, \ldots Z_n$ acting in the line Oz, which (by Art. 23) have a resultant

$$R = \Sigma \,.\, Z \ldots\ldots\ldots (1);$$

and, in the plane xz, a set of negative couples, which (by Art. 52) are equivalent to a single couple in that plane whose moment

$$= -\Sigma\,(Zx);$$

and, in the plane yz, a set of positive couples which are equivalent to one whose moment

$$= \Sigma\,(Zy).$$

If G be the moment of the resultant of these two couples, and θ the angle which its arm makes with Ox, we shall have from Art. 55,

$$G \,.\, \cos\theta = \Sigma\,(Zx), \quad \text{and} \quad G \,.\, \sin\theta = \Sigma\,(Zy);$$

$$\therefore \quad G^2 = (\Sigma \,.\, Zx)^2 + (\Sigma \,.\, Zy)^2 \ldots\ldots\ldots\ldots (2),$$

$$\text{and } \tan\theta = \frac{\Sigma \,.\, Zy}{\Sigma \,.\, Zx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Equation (1) gives the resultant force acting parallel to the original forces at the origin of co-ordinates; and equations (2) (3) give the magnitude of the resultant couple, and the position of its plane.

72. We have determined the position of the *arm* of the resultant couple. That result supposes, as in fig. 17, a negative force acting at that extremity of the arm which is at O, and a

positive force at the other extremity. It will sometimes be more convenient to know the position of the positive axis of the couple. Let α be the inclinations of this axis to the axis of x. Then

$$\cos\alpha = \frac{\Sigma . Zy}{G}, \text{ and } \sin\alpha = -\frac{\Sigma . Zx}{G}.$$

In which equations G is to be accounted positive.

73. The results obtained in Art. 71 are perfectly general, but they admit of reduction to a single force except when R or $\Sigma Z = 0$.

(1) When $\Sigma Z = 0$, there is no force acting at O, and the only resultant is the couple whose moment is G.

(2) When ΣZ is not equal to zero, change the couple whose moment is G into an equivalent couple which has each of its forces $R'R''$ equal to R or ΣZ; its arm will be equal to $\frac{G}{R}$; place this couple as in fig. 18, so that one of its forces R'' balances the resultant R. By this mode the whole are reduced to a single force R' $(=\Sigma Z)$ acting at a point P whose co-ordinates $x'y'$ are known from the equation

$$\begin{aligned} x' &= OP\cos\theta, & \text{and } y' &= OP\sin\theta \\ &= \frac{G}{R}\cos\theta & &= \frac{G}{R}\sin\theta \\ &= \frac{\Sigma . Zx}{\Sigma Z} & &= \frac{\Sigma . Zy}{\Sigma Z}. \end{aligned}$$

These equations are free from ambiguity.

74. *To find the conditions that the system of forces in* Art. 71 *may balance each other.*

We must consider the two cases mentioned in the last Article. In the second of these cases the resultant is the force ΣZ acting at a point whose co-ordinates are

$$\frac{\Sigma . Zx}{\Sigma Z}, \text{ and } \frac{\Sigma . Zy}{\Sigma Z}.$$

There can be no equilibrium as long as this force exists; we must therefore have $\Sigma Z = 0$. But if this be the case the 2nd case coincides with the first; so that the resultant is a couple whose moment is G. There cannot be equilibrium therefore unless $G = 0$; an equation which is equivalent to $\Sigma . Zx = 0$ and $\Sigma . Zy = 0$. Hence the conditions of equilibrium are

$$\Sigma Z = 0, \quad \Sigma . Zx = 0, \quad \Sigma . Zy = 0;$$

which three conditions are both necessary and sufficient.

75. General definition of "*the moment of a force about a line.*"

If the direction of the force be perpendicular to the given line, the moment is equal to the product of the force into the length of a line which is perpendicular *both* to the line in which the force acts and the line about which the moment is required. If the direction of the force be not perpendicular to the given line, it must be resolved into two components, one perpendicular and the other parallel to the given line; the moment of the former will be found by the definition just given, and that of the latter will be zero.

76. According to this definition Z_1y_1 and Z_1x_1 are the moments of Z_1 about the axes of x and y respectively; and hence we may state the three conditions of equilibrium of parallel forces acting on a rigid body as follows:

The sum of all the forces must be equal to zero; and the sums of their moments about any two lines at right angles to each other in a plane which is perpendicular to the direction of the forces must be respectively equal to zero.

77. *To find the conditions of equilibrium of the forces in* Art. 71, *when there is in the body a fixed point; or a fixed line at right angles to the direction of the forces.*

(1) When there is a fixed point.

Let it be taken for the point O in Art. 71; then, as by this arrangement R acts upon an immoveable point, it is not necessary that ΣZ should be $= 0$; but as the couple G would turn the body round O it is necessary and sufficient for equilibrium that $G = 0$, or that

$$\Sigma (Zx) = 0, \quad \text{and } \Sigma (Zy) = 0.$$

That is; *the sums of the moments of the forces about any two lines drawn from the fixed point at right angles to each other, in a plane perpendicular to the direction of the forces, must be separately equal to zero.*

REMARK. In this case, i. e. when there is equilibrium, the pressure on the fixed point is ΣZ acting directly upon it; i. e. it is the same as if all the forces were transposed to that point without altering the direction in which they act.

(2) When there is a fixed line in the body, at right angles to the direction in which the forces act.

Let it be taken for the axis Oy in Art. 71, O being *any* point in it.

Then since the force R acts upon a fixed point it is not necessary for equilibrium that it should be $= 0$. Also the couple G is equivalent to the two $\Sigma (Zx)$, $\Sigma (Zy)$: the latter of which being in the plane yz can be so placed that its forces shall both act upon points in the line Oy, which being immoveable, it is not necessary that this couple should be equal to zero. The remaining couple $\Sigma (Zx)$ tends to turn the body about the fixed line Oy, so that there cannot be equilibrium as long as it exists. Wherefore the only condition which in this case is necessary and sufficient for equilibrium is

$$\Sigma (Zx) = 0,$$

that is, *the sum of the moments of all the forces about the fixed line must be equal to zero.*

REMARK. In this case the pressure on the fixed axis is equivalent to the force ΣZ at O, and the couple $\Sigma (Zy)$; as in Art. 59 (2) these are equivalent to a single force ΣZ acting at a point K in Oy such that

$$OK = \frac{\Sigma (Zy)}{\Sigma Z}.$$

The force ΣZ acting at this point represents the pressure on the axis.

But the pressure may be otherwise represented, for a comparison of the equation $OK = \frac{\Sigma (Zy)}{\Sigma Z}$ with the result found in Art. 59 shews that the pressure on the axis is just the same as if the forces $Z_1 Z_2 Z_3 \ldots$ were transposed to the fixed axis and applied without changing their directions to points in the axis at the respective distances $y_1 y_2 y_3 \ldots$ from O: that is, if through every force we draw planes at right angles to the fixed axis we may transpose each force without altering its direction to the point where the corresponding plane cuts the axis. The forces thus transposed produce the same pressure on the axis as the given system.

78. *To find the resultant of three couples which act upon a rigid body in different planes, no two of which are parallel.*

From any point O (fig. 6) draw three lines OA, OB, OC to represent the axes of the couples, the moments of which are L, M, N; complete the parallelopiped, and join OD, OF. Then the couple whose axis is OD is equivalent to L, M whose axes are OA, OB: and OE is the axis of a couple which is equivalent to OC, OD; i. e. to the three couples L, M, N.

79. *To find the resultant of three couples* L, M, N *whose planes are mutually at right angles.*

From any point O (fig. 7) take OL, OM, ON to represent the axes of the given couples. Then as before we may shew that OP represents the axis of the resultant couple G. Let α, β, γ be the angles POL, POM, PON between the axis of G and the axes of L, M, N. Then since

$$OL = OP\cos\alpha, \qquad OM = OP\cos\beta, \qquad ON = OP\cos\gamma;$$

$$\therefore\ L = G\cos\alpha, \qquad M = G\cos\beta, \qquad N = G\cos\gamma,$$

$$\text{and} \quad \therefore\ L^2 + M^2 + N^2 = G^2.$$

From which four equations the magnitude and position of the resultant couple are known.

80. By means of the four equations just given we may resolve a couple into three components acting in planes at right angles to each other.

81. *To find the resultants of any forces, acting on different points of a rigid body, in lines which are neither parallel nor in one plane.*

Take any point O (fig. 17) of the rigid body as origin, and from it draw any three lines perpendicular to each other for axes of co-ordinates.

Let $x_1 y_1 z_1$, $x_2 y_2 z_2$, ... be the co-ordinates of the points at which the forces act; resolve each force into three components parallel to Ox, Oy, Oz.

Denote the components parallel to x by X_1, X_2, X_3...

.... y by Y_1, Y_2, Y_3...

.... z by Z_1, Z_2, Z_3...

The resultants of the last set of forces we have already found (in Art. 71) to be

a force ΣZ acting at O in the line Oz,

a couple $\Sigma (Zy)$ acting in the plane yz,

and a couple $-\Sigma (Zx)$ acting in the plane xz.

The forces $Y_1 Y_2 Y_3$ ... form a system of parallel forces, of which the resultants may be deduced from those of $Z_1 Z_2 Z_3$... by writing Y, x, z for Z, y, x respectively: they are therefore equivalent to

a force ΣY acting at O in the line Oy,

a couple $\Sigma (Yx)$ acting in the plane xy,

and a couple $-\Sigma (Yz)$ acting in the plane zy.

And in these, writing X, z, y for Y, x, z we find the forces X_1, X_2, X_3 ... equivalent to

a force ΣX acting at O in the line Ox,

a couple $\Sigma (Xz)$ acting in the plane zx,

and a couple $-\Sigma (Xy)$ acting in the plane yx.

Collecting these results it appears that the original forces are equivalent to ΣX, ΣY, ΣZ acting at O; and the three couples

$$\Sigma (Yx) - \Sigma (Xy) = \Sigma (Yx - Xy) \text{ in the plane } xy,$$

$$\Sigma (Xz) - \Sigma (Zx) = \Sigma (Xz - Zx) \text{ in the plane } zx,$$

$$\text{and } \Sigma (Zy) - \Sigma (Yz) = \Sigma (Zy - Yz) \text{ in the plane } yz.$$

Now if R be the resultant of the forces acting at O, and α, β, γ the angles which the line in which it acts makes with Ox, Oy, Oz; and if G be the resultant of the couples, and α', β', γ' the angles which its axis makes with Ox, Oy, Oz, we have by Arts. 34, 79,

$$R \cos \alpha = \Sigma X, \quad R \cos \beta = \Sigma Y, \quad R \cos \gamma = \Sigma Z,$$

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2:$$

$$\text{and } G \cos \alpha' = \Sigma (Zy - Yz) = L \text{ suppose}$$

$$G \cos \beta' = \Sigma (Xz - Zx) = M \text{ ......}$$

$$G \cos \gamma' = \Sigma (Yx - Xy) = N \text{ ......}$$

$$G^2 = L^2 + M^2 + N^2.$$

These eight equations give both the magnitude and direction of the resultant force which acts at the origin of co-ordinates; and the magnitude and position of the axis of the resultant couple. These results are quite general, but we shall now shew that under certain conditions the original forces admit of a single resultant.

82. *To find the condition that the forces in* Art. 81 *may admit of a single resultant, and to find the magnitude and position of it.*

If G be $= 0$, no reduction is necessary; but if not, change the couple G into an equivalent couple, whose forces $R'R''$ are each equal to R; place this couple so that one of its forces (as R') shall act at O, and *if possible* in a direction opposite to R; in this case R and R' balance each other and may be removed; there will then be left only the force R'', which is the same as if the force R had been transposed to R'', and the couple taken away. It appears, then, that a couple and a force are reduced to a single force (where the problem is possible) by taking away the couple and transposing the force to some other point. The possibility of being able to do this, depends on its being possible to place the forces R and R' in the same line. The student will perceive that this can be done only when the force R acts in a line which is perpendicular to the axis of the couple G, the analytical condition of which is

$$0 = \cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma'$$

$$= \frac{\Sigma X}{R} \cdot \frac{L}{G} + \frac{\Sigma Y}{R} \cdot \frac{M}{G} + \frac{\Sigma Z}{R} \cdot \frac{N}{G}.$$

Hence the conditions required are

(1) R must not be $= 0$;

and (2) $L\Sigma X + M\Sigma Y + N\Sigma Z$ must $= 0$.

We have yet to find the line in which the resultant force R'' acts.

We have remarked above, that R'' is the force R transposed without altering its magnitude or direction. If we had begun the investigations of Art. 81 by taking a point in R'' for the origin of co-ordinates, we should have found R'' acting at that origin, and no resultant couple; that is, denoting the co-ordinates referred to this origin by x'', y'', z'' we should have found

$$0 = \Sigma\,(Zy'' - Yz''),$$

$$0 = \Sigma\,(Xz'' - Zx''),$$

$$\text{and } 0 = \Sigma\,(Yx'' - Xy'');$$

these are in fact the conditions that the origin may be a point in the single resultant force. Let x', y', z' be the co-ordinates of this origin referred to the original origin, x, y, z being the same as before; then $x'' = x - x'$, $y'' = y - y'$, $z'' = z - z'$, which being substituted in the above equations give

$$y'\Sigma Z - z'\Sigma Y = \Sigma\,(Zy - Yz) = L,$$
$$z'\Sigma X - x'\Sigma Z = \Sigma\,(Xz - Zx) = M,$$
$$x'\Sigma Y - y'\Sigma X = \Sigma\,(Yx - Xy) = N,$$

x', y', z' are the co-ordinates of any point in the line in which R'' acts. There being *three* equations between these quantities, it would seem as if there existed only a single point at which R'' can be applied, which is contrary to Art. 21: but if we multiply these equations by ΣX, ΣY, ΣZ and add the results we shall find

$$0 = L\Sigma X + M\Sigma Y + N\Sigma Z,$$

which being satisfied by hypothesis, the three equations are not independent, but any one is derivable from the other two. Consequently any two of these are the equations of the line in which the single resultant acts.

83. *When the forces in* Art. 81 *do not admit of being reduced to a single force, they can be reduced to a force and a couple the axis of which is parallel to the force.*

For let ϕ be the angle between the axis of the couple G in Art. 81, and the force R. Resolve G into two components $G\cos\phi$, $G\sin\phi$ whose axes are respectively parallel and perpendicular to R. The latter of these, being compounded with R as in the last Article, will be destroyed, and R will be transposed to some other point of the rigid body, without altering its direction; it is therefore still parallel to the axis of the couple whose moment

$$= G\cos\phi$$
$$= G\,(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma')$$
$$= \frac{L\Sigma X + M\Sigma Y + N\Sigma Z}{R}.$$

This appears to be the simplest form to which the forces in Art. 81 are in general reducible. They may however be presented in another simple form as in the following Article.

84. *The forces in* Art. 81 *can be reduced to two forces acting in two lines which in general do not meet; and to find the shortest distance between these lines,*

Let them be reduced as in the last Article to a force R and the couple $G \cos \phi$.

Let Q (fig. 19) be the point at which R acts; and let the couple $G \cos \phi$ be placed so that one of its forces K acts at Q, PQ being its arm. Then QR being parallel to the axis of the couple is perpendicular to QK; hence if H be the resultant of R and K, and ψ be the angle HQK, the forces are now reduced to K at P, and H at Q, such that

$$H \cos \psi = K = \frac{G \cos \phi}{PQ},$$

$$H \sin \psi = R;$$

$$\therefore\ H^2 = R^2 + \frac{G^2}{PQ^2} \cdot \cos^2 \phi,$$

$$\text{and } \tan \psi = \frac{R \cdot PQ}{G \cos \phi}.$$

Now PQ being at right angles both to QH and PK is the minimum distance between them. It appears from the above equations that PQ is arbitrary; but when it is of given length then both K and H are known, and their relative position from the last equation. Q is known by the preceding Article.

85. *To find the equations of the line in which* R *acts, and of the plane in which* G *acts, in* Art. 81.

Since R passes through the origin its equations are

$$\frac{x'}{\cos \alpha} = \frac{y'}{\cos \beta} = \frac{z'}{\cos \gamma},$$

$$\text{or } \frac{x'}{\Sigma X} = \frac{y'}{\Sigma Y} = \frac{z'}{\Sigma Z}.$$

Again, we may suppose the plane of G to pass through the origin. And since α', β', γ' are the angles which a perpendicular upon it makes with the co-ordinate axes, its equation is

$$x' \cos \alpha' + y' \cos \beta' + z' \cos \gamma' = 0;$$

$$\therefore\ Lx' + My' + Nz' = 0$$

is the equation to the plane in which G acts.

86. The conditions that the plane of G may be perpendicular to the line in which R acts are $\frac{L}{\Sigma X} = \frac{M}{\Sigma Y} = \frac{N}{\Sigma Z}$.

87. *To find the equations of the line in which the resultant force acts when the resultant couple acts in a plane at right angles to it.* (Art. 83).

Let O' be any point in the line in which the resultant acts in this case; x', y', z' its co-ordinates referred to the origin O used in Art. 81. If with the origin O' we were to proceed as in Art. 81, we should find a resultant R acting at O', and a couple G', the plane of which would be found to be perpendicular to the direction of R; and therefore

$$\frac{L'}{\Sigma X} = \frac{M'}{\Sigma Y} = \frac{N'}{\Sigma Z};$$

where L', M', N' represent the quantities

$$\Sigma\,(Zy'' - Yz''),\ \Sigma\,(Xz'' - Zx''),\ \Sigma\,(Yx'' - Xy''),$$

and x'', y'', z'' are the co-ordinates of a point referred to the origin O': hence $x'' = x - x'$, $y'' = y - y'$, $z'' = z - z'$, as in Art. 82;

$$\therefore \frac{\Sigma\,\{Z\,(y - y') - Y\,(z - z')\}}{\Sigma X} = \frac{\Sigma\,\{X\,(z - z') - Z\,(x - x')\}}{\Sigma Y}$$

$$= \frac{\Sigma\,\{Y\,(x - x') - X\,(y - y')\}}{\Sigma Z},$$

or bringing x', y', z' outside of the symbol Σ, and writing L, M, N for their equals, the equations required are

$$z'\frac{\Sigma Y}{\Sigma X} - y'\frac{\Sigma Z}{\Sigma X} + \frac{L}{\Sigma X} = x'\frac{\Sigma Z}{\Sigma Y} - z'\frac{\Sigma X}{\Sigma Y} + \frac{M}{\Sigma Y}$$

$$= y'\frac{\Sigma X}{\Sigma Z} - x'\frac{\Sigma Y}{\Sigma Z} + \frac{N}{\Sigma Z}.$$

88. In Art. 83 we were able, by transposing R to *destroy* the couple $G \sin \phi$. If afterwards we transpose R to some other point, we shall thereby *introduce* a new couple, the axis of which being at right angles to the axis of the couple $G \cos \phi$ would be compounded with it, and make a resultant couple greater than $G \cos \phi$. Hence to whatever point R be transposed, the resultant couple will always be greater than in Art. 83. Consequently the resultant couple is a *minimum* when its axis is parallel to the resultant force. This is sometimes called the *principal* couple.

89. Def. The line in which R acts when the resultant couple is a minimum, is called the *central axis*. Its equations are found above in Art. 87.

90. If R be transposed from the central axis to a distance a from it, a couple is thereby introduced whose arm is a and moment Ra; consequently the resultant couple for this position of R is $\sqrt{R^2a^2 + G^2 \cos^2 \phi}$, which is constant as long as a is constant. Hence if we construct a cylindrical surface having the central axis for its axis, the surface of this cylinder will be the locus of the origins, which will give equal resultant couples.

91. *To find the conditions that the forces in* Art. 81 *may balance each other when the body upon which they act is free.*

(1) Suppose the direction of R to be not parallel to the plane in which G acts; then since R and G cannot in this case be reduced to a single force, they must be separately equal to zero;

$$\therefore\ 0 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$

$$\text{and } 0 = L^2 + M^2 + N^2;$$

which are equivalent to the six following:

$$0 = \Sigma X, \qquad 0 = \Sigma Y, \qquad 0 = \Sigma Z,$$

$$0 = L, \qquad 0 = M, \qquad 0 = N.$$

(2) Suppose the direction of R to be parallel to the plane in which G acts; then R and G can be reduced to a single force, the effect of which reduction is to transpose R and destroy G. There can therefore be no equilibrium unless $R=0$; it is *necessary* therefore that R should be $=0$. But if $R=0$, R and G cannot be reduced to a single force; that is, G cannot be destroyed by transposing R; it is therefore also *necessary* that G should separately be $=0$. Hence the conditions of equilibrium are the same in this as the preceding case; and, observing that L, M, N are the moments of the forces about the lines Ox, Oy, Oz, we may thus state them in words:

The sums of the resolved parts of the forces parallel to any three lines at right angles to each other must be separately equal to zero; and the sums of the moments of the forces, about any three lines at right angles to each other and passing through a point, must be separately equal to zero.

92. *To find the conditions that the forces in* Art. 81 *may balance each other when one point of the rigid body is fixed.*

Let this point be taken for the point O in Art. 81. Then since by this arrangement R acts upon a fixed point, it is not necessary for equilibrium that R should vanish; but as the couple G would turn the body about this point, it is necessary and sufficient for equilibrium that G be $=0$; that is, that

$$L=0, \quad M=0, \quad N=0.$$

Or, in words: *The sums of the moments of the forces, about any three lines at right angles to each other passing through the fixed point, must be separately equal to zero.*

REMARK. The pressure on the fixed point is represented by R acting directly upon it: i. e. it is the same as if all the forces of the system were transposed to the fixed point without changing their directions.

93. *To find the conditions that the forces in* Art. 81 *may balance each other when there is in the body a fixed axis.*

Let the fixed axis be taken as the axis of z in Art. 81, and any point in it as the point O; then since R acts upon a fixed line it is not necessary for equilibrium that R should be equal to zero; also the couples L, M, acting in the planes yz, xz, can be turned round and so placed that their forces shall all act upon the fixed line Oz; but the couple N acting in the plane xy cannot be so placed, and therefore as long as it exists it will turn the body round the line Oz; consequently it is necessary and sufficient for equilibrium that $N=0$;—or in words,

The sum of the moments of the forces about the fixed axis must be equal to zero.

REMARK. The pressure on the fixed axis is represented by the force R at the origin and the forces of the two couples L, M applied directly to the axis. But R is equivalent to the three ΣX, ΣY, ΣZ; of which ΣX can be compounded with the couple M (the forces of which are in the same plane with it) as in Art. 59 (2); the result of this compounding is a single force ΣX acting at a point in the axis the abscissa of which is $\frac{L}{\Sigma X}$. The force ΣY may in like manner be compounded with the couple M; and the result in this case is ΣY acting at the point $-\frac{M}{\Sigma Y}$. Hence then the pressures on the axis are represented by

$$\left.\begin{array}{l} \Sigma X \left(\text{at the point } \dfrac{L}{\Sigma X}\right) \text{ parallel to } x, \\ \Sigma Y \left(\text{............} -\dfrac{M}{\Sigma Y}\right) \text{ parallel to } y, \end{array}\right\} \text{.........} (A),$$

$$\Sigma Z \text{ (at any point of the axis) parallel to } z.$$

The last of these (ΣZ) may be compounded with either of the others; and thus in the most general case the pressure on a fixed axis may be represented by two forces.

COR. The pressure ΣZ urges the axis in the direction of its length, the other two pressures ΣX, ΣY can be reduced to a force and a couple, the plane of the couple being perpendicular to the direction of the force. As this reduction is useful in certain cases, we shall shew how it may be effected.

Through the fixed axis draw a plane so inclined to the two forces ΣX, ΣY that the resolved parts of ΣX, ΣY along this plane may be *equal* and in *contrary* directions. Let a normal to this plane make an angle θ with ΣX, and therefore an angle $90^\circ - \theta$ with ΣY; then the components

$$\text{of } \Sigma X \text{ are } \begin{cases} \Sigma X \,.\, \cos\theta \text{ parallel to the normal,} \\ \Sigma X \,.\, \sin\theta \text{ along the plane;} \end{cases}$$

$$\text{of } \Sigma Y \text{ are } \begin{cases} \Sigma Y \,.\, \sin\theta \text{ parallel to the normal,} \\ \Sigma Y \,.\, \cos\theta \text{ along the plane;} \end{cases}$$

of which four components $\Sigma X \,.\, \sin\theta$ is equal to $\Sigma Y \,.\, \cos\theta$ by hypothesis; and therefore these two form a couple, and fix the value of θ; for since

$$\Sigma X \,.\, \sin\theta = \Sigma Y \,.\, \cos\theta,$$

$$\therefore \tan\theta = \frac{\Sigma Y}{\Sigma X};$$

the arm of this couple is

$$= \frac{L}{\Sigma X} + \frac{M}{\Sigma Y}, \text{ from } (A),$$

and therefore its moment is

$$= \frac{L}{\Sigma X} \,.\, \Sigma X \,.\, \sin\theta + \frac{M}{\Sigma Y} \,.\, \Sigma Y \,.\, \cos\theta$$

$$= \frac{L\Sigma Y + M\Sigma X}{\{(\Sigma X)^2 + (\Sigma Y)^2\}^{\frac{1}{2}}}.$$

And the positive axis of this couple is inclined to the axis of x at the angle θ, given above.

Of the four components mentioned above, the two not yet reduced are $\Sigma X \,.\, \cos\theta$, $\Sigma Y \,.\, \sin\theta$, acting in one plane on the points $\frac{L}{\Sigma X}$, $-\frac{M}{\Sigma Y}$. They are therefore (Art. 40 or 59) equivalent to a single force

$$= \Sigma X \,.\, \cos\theta + \Sigma Y \,.\, \sin\theta = \{(\Sigma X)^2 + (\Sigma Y)^2\}^{\frac{1}{2}},$$

acting at an inclination θ to the axis of x, upon a point in the fixed axis the distance of which from the origin is (by Art. 59)

$$=\frac{\Sigma X.\cos\theta.\dfrac{L}{\Sigma X}-\Sigma Y.\sin\theta.\dfrac{M}{\Sigma Y}}{\Sigma X.\cos\theta+\Sigma Y.\sin\theta}=\frac{L\Sigma X-M\Sigma Y}{(\Sigma X)^2+(\Sigma Y)^2}.$$

94. *To find the conditions that the forces in* Art. 81 *may balance each other, when there is in the body a line moveable lengthwise but in no other direction.*

Let this line be taken as the axis of z in Art. 81, and any point in it as the point O; then R acts upon this line, and being resolved into its components ΣX, ΣY, ΣZ, the first two acting in directions in which the line cannot move produce no effect; but ΣZ acting in the direction in which the line can move must be equal to zero if there be equilibrium. Also the couples L, M, being turned round and so placed that their forces shall act upon the line Oz, produce no effect because they urge it in directions in which by hypothesis it cannot move: but the couple N cannot be so placed, and therefore as long as it exists it will turn the body about the line Oz; it is therefore necessary that N should be equal to zero. Hence the conditions necessary and sufficient for equilibrium in this case are,

The sum of the resolved parts of the forces parallel to the given line must be equal to zero; and the sum of the moments of the forces about the same line must also be equal to zero.

This Art. will be applied when we come to investigate the power of a screw.

95. The preceding Articles have been enunciated for *rigid* bodies only: but since when a *flexible* body or a body that has joints is in equilibrium it may be supposed to become rigid without affecting its equilibrium, all the conditions of equilibrium before investigated must be satisfied by a flexible body or a body that has joints. But it is to be noticed particularly that all these conditions may be fulfilled and yet such a body not be in equilibrium, for some of its parts may not be in equilibrium.

As a simple instance take the following. A straight rod placed in a horizontal position with its ends on two props will be in equilibrium; but a chain, or a rod with a joint in the middle, so placed would fall. Hence then in equilibrium flexible and jointed bodies satisfy all the conditions which rigid bodies satisfy; and besides them such other conditions as are necessary to secure the equilibrium of every part into which they are divided by joints: the actions at each joint are, though unknown generally, equal and opposite upon the two parts joined there.

96. *If three forces acting upon a rigid body balance each other, the lines in which they act must be in one plane, and either be parallel or pass through a point.*

When a rigid body is in equilibrium, we may suppose any line or point in it to become fixed without affecting the equilibrium: upon this principle let an axis, not parallel to any of the forces and intersecting the lines in which *two* of the given forces act, become fixed; then these two forces acting upon fixed points may be removed; which being done the body having a fixed axis is kept in equilibrium by the remaining force, which is impossible unless the line in which this force acts either intersect the fixed axis, or be parallel to it. But it is not parallel to it by hypothesis, therefore it intersects it. It appears then that *any* axis, not parallel to one of the forces, and intersecting two of them, must meet the directions of all the forces, consequently they are all in one plane. Again, since they are all in one plane they must either be all parallel, or some two of them must intersect; in the latter case, the point of intersection may be supposed to become fixed, and the corresponding forces removed; and then the rigid body having a fixed point is kept in equilibrium by a single force, which is impossible unless its direction pass through the fixed point; consequently, the directions of all the forces either are parallel or pass through a point.

97. The student will have remarked that when forces (as in Chaps. I. II.) act on a point, it is not a necessary condition of equilibrium that their *moments* about an axis should be equated to zero. The same is true of every system which is

capable of being reduced to forces acting on a point. Also if in any case of equilibrium we know that the forces are capable of being reduced to three forces not parallel, since these by the last Art. must act in lines passing through a point, the same is true.

98. In such of the preceding Articles as relate to the conditions of equilibrium of a rigid body under the action of a system of forces, the lines parallel to which the forces are to be resolved, or about which the moments are to be taken, and equated to zero, have been spoken of as necessarily perpendicular to each other. This necessity, however, has entirely arisen from the mode in which we have conducted our investigations; from our having, in fact, assumed the co-ordinate axes to be rectangular. We shall shew that it may be dispensed with; and that it is sufficient if the forces be resolved in directions of, and the moments taken about, any three lines, provided no two of them are parallel, and all three not in the same plane. For this purpose it will be necessary to prove the following propositions.

99. *If from a point there be drawn three lines not in one plane, and the sums of the components, parallel to them, of all the forces be separately equal to zero; and also the sums of the moments of all the forces about them be separately equal to zero; there will be equilibrium.*

For from the proposed point let there be drawn a system of three rectangular co-ordinate axes Ox, Oy, Oz; and let one of the proposed lines make angles ξ_1, η_1, ζ_1 with them. Then the sum of the components of the forces in the direction of this line is

$$\Sigma (X \cos \xi_1) + \Sigma (Y \cos \eta_1) + \Sigma (Z \cos \zeta_1),$$

which is equal to

$$\Sigma X . \cos \xi_1 + \Sigma Y . \cos \eta_1 + \Sigma Z . \cos \zeta_1:$$

and therefore by hypothesis

$$0 = \Sigma X . \cos \xi_1 + \Sigma Y . \cos \eta_1 + \Sigma Z . \cos \zeta_1.$$

Or, if R be the resultant of ΣX, ΣY, ΣZ; and α, β, γ the angles which its direction makes with the co-ordinate axes;

$$\begin{aligned} 0 &= R\cos\alpha \,.\, \cos\xi_1 + R\cos\beta\cos\eta_1 + R\cos\gamma\cos\zeta_1 \\ &= R\cos a. \end{aligned}$$

Similarly, $0 = R\cos b$,

and $0 = R\cos c$;

where a, b, c, are the angles which the direction of R makes with the three proposed lines. Now these three equations require either that R should be $=0$, or that $\cos a$, $\cos b$, $\cos c$ should *each* be $=0$; but this last supposition is impossible, because the given lines are *not all* in one plane by hypothesis;

$$\therefore\ R = 0.$$

Again, the couples L, M, N have their axes parallel to Ox, Oy, Oz respectively: hence resolving them each into two components, one of which has its axis parallel to the line ξ_1, η_1, ζ_1, and the other has its axis perpendicular to it, we have the sum of the former $= L\cos\xi_1 + M\cos\eta_1 + N\cos\zeta_1$, this, being the couple which tends to turn the body about the line under consideration, is the moment of all the forces about that line, and therefore by hypothesis

$$\begin{aligned} 0 &= L\cos\xi_1 + M\cos\eta_1 + N\cos\zeta_1 \\ &= G\,.\cos\alpha'\cos\xi_1 + G\cos\beta'\cos\eta_1 + G\cos\gamma'\cos\zeta_1 \\ &= G\cos a'. \end{aligned}$$

Similarly $0 = G\cos b'$,

and $0 = G\cos c'$,

a', b', c' being the angles which the axis of G the resultant of L, M, N makes with the three proposed lines. From these three equations it follows as before, that $G=0$; and we have already shewn that $R=0$; consequently there is equilibrium.

100. Cor. If there be drawn three lines not in one plane, no two of which are parallel, and the sums of the components, parallel to them, of all the forces be equal to zero; then the resultant R is equal to zero. For the first part of the preceding

demonstration applies here, since it does not depend upon the positions of the lines, but only on their directions.

101. *If there be three lines not in one plane, no two of which are parallel, and the sums of the components, parallel to them, of all the forces be separately equal to zero; and if there be three lines (not necessarily the same as the former) not in one plane, no two of which are parallel, and the sums of the moments of all the forces about them be separately equal to zero; there will be equilibrium.*

The demonstration contained in the former part of Art. 99, does not depend at all upon the three lines being drawn from a point as required in the proposition, and therefore it will apply here; consequently $R=0$. From this it follows, that if our present system of forces be not in equilibrium, their resultant is a couple, G suppose. Let a', b', c' be the angles which the axis of G makes with the three lines mentioned in the latter part of our proposition; then resolving G into two components, the axis of one being parallel, and that of the other perpendicular to the first of the three lines, we have the moment of all the forces about that line (which is equal to the former component couple)

$$= G \cos a',$$

which by hypothesis is equal to zero. Hence

$$0 = G \cos a'.$$

$$\text{Similarly } 0 = G \cos b',$$

$$\text{and } 0 = G \cos c'.$$

Consequently $G=0$; and we have already shewn that $R=0$; therefore there is equilibrium.

CHAPTER V.

ON THE PRINCIPLE OF VIRTUAL VELOCITIES.

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102. DEF. IF the parts of a rigid body, or of a system of rigid bodies, in equilibrium, be geometrically transferred through a very small space in any manner, the space moved over by any particle is called, in Statics, the *velocity* of that particle.

The path described by any particle is supposed to be so small, that it may in every case be taken as a straight line, on the principle that an arc of a curve ultimately coincides with its chord.

The velocity of a point, estimated in the direction of the line in which the force acted upon the point when the body was in its position of equilibrium, is called the *virtual velocity* of the point.

103. *Having given the velocity of a point, to estimate its velocity in any proposed direction in the plane of motion.*

Let AB (fig. 20) be the velocity of a point, EF the direction in which it is required to estimate it. Draw EG perpendicular to EF; Aa, Bb parallel to EF; and AC parallel to EG. Then every line perpendicular to EG in the plane FEG is parallel to, and therefore in the same direction as EF. Hence, to find the velocity in the direction EF, is the same as to find the space through which the point has receded from the line EG. Now at A the distance from EG is Aa, and at B the distance is Bb, consequently the velocity estimated in the direction EF is

$Bb - Aa = BC = AB \cos ABC$. And ABC is equal to the angle which the velocity AB makes with the proposed direction.

Hence we can estimate a velocity in a proposed direction, by multiplying the velocity into the cosine of the angle at which it is inclined to the proposed direction.

104. From this it will be seen, that when a particle, which is acted on by a force, is displaced, the virtual velocity of that particle will be found as follows;—drop a perpendicular from the new position of the particle upon the line in which the force acted before displacement, and the line intercepted between the foot of this perpendicular and the first position of the point, is the virtual velocity required. Thus, in fig. 21, let the force F act upon the point A in the direction AF, and let A be moved to A'; draw $A'a'$ perpendicular to AF, then Aa' is the virtual velocity of A. If A were moved to A'' so that FAA'' is a right angle, the virtual velocity of A would be zero. If A were moved to A''' so that the perpendicular $A'''a'''$ falls on FA *produced*, the virtual velocity Aa''' of A is said to be *negative*.

105. *If a rigid body be displaced in any manner, the velocities of any two of its particles, estimated in the direction of the line which joins them, are equal.*

Let A, B (fig. 22) be two particles of a rigid body, and let AA', BB' be their velocities. Then, because the body is rigid, $A'B' = AB$. Through A draw a plane at right angles to AB, and upon it drop the perpendiculars $A'a$, $B'b$. It will be easily seen, that the estimated velocity of A is $A'a$; and that of B, $B'b - BA$; and we are to prove these equal. Join ab, and draw $A'C$ parallel to it. The angles at C are right angles, and therefore

$$\begin{aligned} B'b - BA &= A'a + B'C - BA \\ &= A'a + A'B' \cos A'B'C - BA \\ &= A'a - BA\,(1 - \cos A'B'C) \\ &= A'a - 2BA \,.\, \sin^2 \frac{A'B'C}{2}. \end{aligned}$$

But the last term, containing the square of the very small quantity $\sin \frac{A'B'C}{2}$ as a factor, must be omitted in conformity with our definition in Art. 102.

Hence $B'b - BA = A'a$.

This proposition is true, if A, B be two particles of different bodies connected by a rigid rod, or inextensible string; for in the preceding demonstration nothing more is assumed than that $A'B'$ is equal to AB, which is satisfied in these cases.

106. If the reader should have any doubt respecting the propriety of omitting the last term, we would recommend him to reconsider the consequences of the definition in Art. 102, where it is stated that the displacement of every particle is so small, that curve lines may be considered as coinciding with their chords; this requires us to consider the deflection of an arc from its tangent as evanescent in comparison of the arc itself, which arc is the velocity with which we have to deal. Hence

$$BA(1 - \cos A'B'C),$$

being the versed sine (or deflection from the tangent) of the arc which represents a quantity less than the velocity, may be *a fortiori* neglected.

107. If the displacements of the two points in Art. 102 be in *parallel* straight lines through *finite* spaces, the proposition of Art. 105 will then also be *accurately* true; and our definitions in Art. 102, and the property in Art. 103, will also strictly hold, how large soever be the spaces through which the particles are displaced.

108. If the particles A, B, in Art. 105, are urged by two equal forces T, T' in opposite directions along the line AB, the virtual velocities δs, $\delta s'$ of A, B for those forces will be equal, but of contrary signs: and consequently the quantity $T\delta s + T'\delta s'$ is equal to zero. Now if A, B be two particles of the same rigid body (or of two different bodies connected in such a manner by a rod or cord AB that the distance between

them does not change), their influence upon each other is exerted along the line AB, and is called tension. This tension is the same for both, but acts upon them in opposite directions, viz. either to draw them towards each other, or to push them asunder. Hence it follows, that for the tensions acting between A and B, $T\delta s + T'\delta s' = 0$. The same may be proved for any and every two points in a whole system of bodies, provided they are connected in such a manner, that the distance between the points of connection is not changed by the displacement. It is obvious, that the tensions we are now considering, occur in pairs. Hence it follows, that *if the forces of tension throughout a whole system of bodies in equilibrium be respectively multiplied by the virtual velocities of the points on which those tensions are supposed to act, the sum will be equal to zero.*

109. If a body rest against a smooth fixed point, there will be a pressure of the point against the body in the direction of a normal to the surface of the body. This pressure is one of the forces which keep the body in equilibrium. Let A (fig. 23) be the fixed point, PA the surface of the body resting against it, AR a normal at A, and let the body be displaced *without lifting it off the point*, so that A comes to some point A' suppose. Then the virtual velocity is

$$AA' \cos RAA' = AA' \sin PAA';$$

but PAA' is an indefinitely small angle, and therefore $AA' \sin PAA'$ is indefinitely smaller than AA', and may be neglected. Hence, if R be multiplied into its virtual velocity, the product may be neglected.

110. If a body rest against a smooth fixed curve line or surface, there will be a pressure of the curve or surface against the body, in the direction of a normal at the point of contact.

Let PA (fig. 24) be the body resting against the curve line or surface QA; and let the body, by sliding and rolling, come into the position $P'BA'$, B being now the point of contact, and A' the new position of A. The virtual velocity of $A = BA' \sin ABA'$; which, for the same reason as before, may be neglected.

111. If two *smooth* bodies of a system rest against each other, there will be a mutual pressure, which will act upon them at the point of contact in opposite directions, coinciding with the common normal at that point. If they are disturbed *without being separated*, the distance between their centres of curvature, at the point of contact, will remain unchanged; and, therefore, the virtual velocities will be exactly equal, but of contrary signs for the two bodies. If, then, R, R' be the equal pressures exerted by each against the other, and δr, $\delta r'$ the virtual velocities,

$$R\delta r + R'\delta r' = 0.$$

112. From the last three Articles, it appears that in any system of bodies kept in equilibrium by the action of external forces, and by tensions, reactions of smooth fixed obstacles, and mutual pressures of smooth bodies of the system, the sum of the products of each tension, reaction, and pressure, into the corresponding virtual velocity of the point on which it acts, is equal to zero.

The student will remark, that the Articles referred to, are only true when the displacement of the system is so small as to agree with the definition of a velocity given in Art. 102: also, in the case of pressures, the surfaces must be smooth, and the contact must not be broken; and in the case of tensions, the connecting line must be of unaltered length.

113. *Let there be any number of connected bodies of a system kept in equilibrium by the action of external forces, and also by the tensions of connecting rods, cords, &c., by the reactions of smooth fixed obstacles, and by mutual pressure of smooth parts; then, if each external force be multiplied into the virtual velocity of the point on which it acts, the sum of all such products for the whole system is equal to zero.* It is necessary (as the reader will see from the preceding Articles), in geometrically displacing the system, that no contacts be broken, and that rods and cords remain of the same length as in the equilibrium position. This is the principle of virtual velocities.

If a particle of one of the bodies be acted on by the external forces F_1, F_2,...F_p and by the tensions T_1, T_2,... T_q, and the reactions and pressures R_1, R_2,...R_u, we may consider that point as free, and kept in equilibrium by the action of all these sets of forces.

Let $\alpha_1\beta_1\gamma_1$ $\alpha_2\beta_2\gamma_2$... $\alpha_p\beta_p\gamma_p$ be the angles which the forces F_1, F_2...F_p make with the co-ordinate axes; $a_1b_1c_1$, $a_2b_2c_2$...$a_qb_qc_q$ and $a'_1b'_1c'_1$, $a'_2b'_2c'_2$...$a'_ub'_uc'_u$ similar quantities for T_1, T_2,... T_q and R_1, R_2,... R_u.

Then, because the particle is in equilibrium under the action of these forces, therefore (Art. 38)

$$0 = \Sigma\,(F\cos\alpha) + \Sigma\,(T\cos a) + \Sigma\,(R\cos a'),$$

$$0 = \Sigma\,(F\cos\beta) + \Sigma\,(T\cos b) + \Sigma\,(R\cos b'),$$

$$0 = \Sigma\,(F\cos\gamma) + \Sigma\,(T\cos c) + \Sigma\,(R\cos c').$$

Let now the system be displaced, the velocity of the particle under consideration being δS_1, and $\xi\eta\zeta$ the angles which δS_1 makes with the co-ordinate axes. Then, if δs_1 be the virtual velocity for the force F_1,

$$F_1\delta s_1 = F_1\,(\cos\alpha_1\cos\xi + \cos\beta_1\cos\eta + \cos\gamma_1\cos\zeta)\,\delta S_1.$$

Similar expressions are true for the other external forces which act on the point, and therefore

$$\Sigma\,(F\delta s) = \{\Sigma\,(F\cos\alpha)\cos\xi + \Sigma\,(F\cos\beta)\cos\eta + \Sigma\,(F\cos\gamma)\cos\zeta\}\,\delta S_1.$$

Similarly, if δt_1 and δr_1 be the virtual velocities corresponding to T_1 and R_1,

$$\Sigma\,(T\delta t) = \{\Sigma\,(T\cos a)\cos\xi + \Sigma\,(T\cos b)\cos\eta + \Sigma\,(T\cos c)\cos\zeta\}\delta S_1;$$

and

$$\Sigma\,(R\delta r) = \{\Sigma\,(R\cos a')\cos\xi + \Sigma\,(R\cos b')\cos\eta + \Sigma\,(R\cos c')\cos\zeta\}\,\delta S_1.$$

Hence by adding the last three equations we obtain

$$\Sigma\,(F\delta s) + \Sigma\,(T\delta t) + \Sigma\,(R\delta r) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which the symbol Σ extends to all the forces, tensions, and reactions which act upon the point under consideration, but has no reference to the other particles of the system.

We may form equations similar to (1) for every other point in the whole system upon which forces of any kind whatsoever act. If all these be added together, the terms belonging to the tensions along the lines which join points of the same body, and also those which act along rods and cords connecting two points of separate bodies of the system; and likewise the reactions of fixed points, and surfaces, and the mutual pressures of two bodies of the system, will all disappear, by Art. 112, in forming the sum. But these, together with the external forces, are all the forces which act on the system; consequently, there remains only the equation

$$\Sigma' (\Sigma \cdot F\delta s) = 0;$$

where Σ' extends to all the points of the system upon which external forces act, Σ' and Σ together denote that the sum of the products of all the external forces which act upon all the points of the system into their respective virtual velocities is to be taken, and the equation shews that this sum is equal to zero; which is the principle of virtual velocities. It is not necessary to employ both Σ and Σ', if we suppose Σ to extend over the whole system, the equation may be written

$$\Sigma (F\delta s) = 0,$$

which is called the *equation of Virtual Velocities*.

114. The great advantage of the equation of virtual velocities consists in this, that it furnishes at once a relation among the external forces which act upon a system, free from tensions and pressures. Since the bodies are rigid, and supposed to be connected by strings or rods of unchangeable length, it is obvious that, in general, when one part is arbitrarily disturbed, the disturbance of the other parts will depend upon it by geometrical relations. In this case, δs_1 being given, δs_2, δs_3 ... will be determinable in terms of δs_1; and these values being written in the equation $\Sigma (F\delta s) = 0$, will give only *one* relation among the forces, and will not therefore enable us to find the forces themselves, if their number exceed two.

It will, however, sometimes be possible to disturb one part of the system without affecting other parts; or the system may consist of several parts, each one of which it may be possible to disturb in such a manner as not to affect the other parts. In this case it is manifest, that the equation of virtual velocities will furnish us as many equations between the forces, as there are parts of the system which can be independently disturbed. Now two points can be independently disturbed when no geometrical relation exists between their virtual velocities. Wherefore, in using the equation $\Sigma(F\delta s)=0$, we must find, from the geometrical properties of the system, as many of the quantities δs_1, δs_2, δs_3... in terms of the others as possible, and substitute them in the equation; the virtual velocities which are still left in it are independent, because no geometrical relation exists among them; and, therefore, the corresponding parts of the system admit of independent disturbance; we must consequently equate the coefficients of each of these terms to zero. The resulting equations are the conditions of equilibrium.

To illustrate what is here meant, we will solve the two following problems by the principle of virtual velocities.

115. *A particle rests upon a plane curve line, being acted on by two forces* X, Y *parallel to the co-ordinate axes: to find the conditions of equilibrium.*

Let $y=f(x)$ be the equation of the curve, x, y being the co-ordinates of the position of equilibrium of the particle. Then since after the disturbance the particle still remains upon the curve, if $y+\delta y$, and $x+\delta x$ be the co-ordinates of its new position they must satisfy the equation of the curve;

$$\therefore\ y+\delta y=f(x+\delta x)=y+d_x y\,.\,\delta x;$$

$$\therefore\ \delta y=d_x y\,.\,\delta x.$$

Now δx, δy are the virtual velocities of the particle for the two forces X, Y;

$$\therefore\ X\delta x+Y\delta y=0 \text{ by the principle};$$

$$\therefore\ X\delta x+Yd_x y\delta x=0 \text{ for all values of } \delta x,$$

$$\text{and } \therefore\ X+Yd_x y=0,$$

which is the condition of equilibrium.

116. *A particle rests upon a smooth curve surface acted on by three forces* X, Y, Z *parallel to the co-ordinate axes: to find the conditions of equilibrium.*

Let $z = f(x, y)$ be the equation of the curve surface, x, y, z being the co-ordinates of the position of equilibrium of the particle. Then if $x + \delta x$, $y + \delta y$, $z + \delta z$ be the co-ordinates of the position of the particle after disturbance, δx, δy, δz are the virtual velocities of the particle for the forces X, Y, Z respectively; and therefore by the principle of virtual velocities,

$$X\delta x + Y\delta y + Z\delta z = 0.$$

But because $x + \delta x$, $y + \delta y$, $z + \delta z$ are the co-ordinates of a point in the curve,

$$\begin{aligned} z + \delta z &= f(x + \delta x, \; y + \delta y) \\ &= z + d_x z \,.\, \delta x + d_y z \,.\, \delta y; \end{aligned}$$

$$\therefore \; \delta z = d_x z \,.\, \delta x + d_y z \,.\, \delta y.$$

By substituting this value of δx, we have

$$(X + Zd_x z)\, \delta x + (Y + Zd_y z)\, \delta y = 0.$$

There is no geometrical relation existing between δy and δx; consequently, the equations of equilibrium are

$$X + Zd_x z = 0, \quad Y + Zd_y z = 0.$$

117. *If two forces* P, P′ *whose virtual velocities are* δp, δp′, *act upon a rigid body at different points, and be such that the equation* $\mathrm{P}\delta\mathrm{p} + \mathrm{P}'\delta\mathrm{p}' = 0$ *is true for all arbitrary displacements of the body, then* P *and* P′ *are equal and act in the same line in opposite directions.*

The equation shews that δp and $\delta p'$ are always zero together. Now disturb the body in such a way that the point at which P acts may remain stationary; then since the body is rigid, the point on which P' acts must have described a circular arc about the stationary point; and as $\delta p' = 0$, that arc must be perpendicular to the direction in which P' acts, therefore P' acts in the direction of a normal to the arc, i.e. in a line passing through the point on which P acts. In the same way it may be shewn that P acts in a line passing through the point at which P' acts; hence they both act in the same line: it will therefore be possible to disturb the body so that δp and $\delta p'$ may be equal in magnitude; and they must have different algebraic signs ($\because P\delta p + P'\delta p' = 0$), which can only happen, since the body is rigid, by reason of P and P' acting in opposite directions; and therefore P and P' are likewise equal.

118. *If the equation* $\Sigma\,(F\delta s) = 0$ *be true for all arbitrary displacements of a rigid body under the action of external forces* F_1, F_2 ... *there is equilibrium.*

For if not, there will be at most two resultants (Art. 84); apply forces P, P' equal to these resultants and in the contrary directions to them, and then the body is in equilibrium under the action of the forces F_1, F_2 ... P, P'; consequently by the Principle of Virtual Velocities,

$$\Sigma\,(F\delta s) + P\delta p + P'\delta p' = 0.$$

But $\Sigma\,(F\delta s) = 0$ by hypothesis, and therefore $P\delta p + P'\delta p' = 0$: and hence it follows from the last article that P and P' are equal and act in opposite directions; consequently they destroy each other; they may therefore be removed without affecting the equilibrium; hence the body is in equilibrium when F_1, F_2, F_3 ... are the only external forces which act on the body.

119. When a system of connected bodies is in equilibrium under the action of external forces, pressures, &c., the equilibrium would not be affected if the connecting joints, cords, &c. were all to become rigid: and hence any force may be transmitted to any point of the system in the line of its action (Art. 21), provided the original point and the new point of application are not situated in independent parts of the system.

120. *If the equation* $\Sigma\,(F\delta s) = 0$ *be true for all arbitrary displacements of a system of connected rigid bodies, there is equilibrium.*

If the system consist of independent parts, let one of those parts alone be displaced, then for that part $\Sigma\,(F\delta s) = 0$ by hypothesis. If that part is not in equilibrium we may apply forces to each body of it which shall keep each of them in equilibrium: these forces (Art. 119) may be transmitted and reduced to two,

P, P', acting upon the part under consideration. Hence reasoning as in Art. 118, we find P and P' equal and opposite, and therefore they may be removed without disturbing the equilibrium of the part. The same may be proved of each of the independent parts; and, consequently, the whole system is in equilibrium.

REMARK. We have seen that the principle of virtual velocities is true only when the displacements are so small as to allow us to consider an arc as coincident with its chord or tangent. Now the reader who is familiar with the Differential Calculus will know, that an arc and its tangent coincide analytically only as far as the second term of Taylor's theorem inclusive: hence the principle of virtual velocities embraces only quantities of the first order of smallness. The second term of Taylor's theorem has been called the *differential* of the first term; wherefore, in applying the principle of virtual velocities, we ought always to use ds instead of δs. The equation of virtual velocities in its proper form is $\Sigma\,(Fds) = 0$. Also because this equation involves only differentials of the first order, it is a matter of indifference whether a body rest upon a curve or its tangent, a surface or its tangent plane; or *on any* other curve or surface having the same tangent or tangent plane at the point on which it rests.

CHAPTER VI.

ON THE CENTRE OF PARALLEL FORCES, AND ON THE CENTRE OF GRAVITY.

THE CENTRE OF PARALLEL FORCES.

121. *If a rigid body be acted on at different points by forces in parallel directions, there is a certain point through which their resultant passes, whatever be the position of the body with respect to the direction in which the forces act.*

Let F_1, $F_2 \dots F_n$ act on the points A, $B \dots K$ (fig. 25) of a rigid body. From any point O in the body draw the rectangular co-ordinate axes Ox, Oy, Oz. Join A, B; and let the resultant of F_1, F_2, pass through C. Draw Aa, Bb, Cc parallel to Oz; join a, b passing through c.

Let $x_1y_1z_1$, $x_2y_2z_2 \dots x_ny_nz_n$ be the co-ordinates of the points on which the forces act; $x'y'z'$ those of C; and let θ be the inclination of AB to ab. Then

$$z' - z_1 = Cc - Aa = AC \sin\theta,$$

$$\text{and } z_2 - z' = Bb - Cc = BC \sin\theta;$$

$$\therefore \frac{z_2 - z'}{z' - z_1} = \frac{BC}{AC} = \frac{F_1}{F_2} \text{ (by Art. 40)};$$

whence we find $(F_1 + F_2)\, z' = F_1 z_1 + F_2 z_2$.

Again, take away the forces F_1, F_2 and replace them by their resultant $F_1 + F_2$ acting at C; then if we put $x''y''z''$ for the co-ordinates of the point through which the resultant of F_1, F_2, F_3, or, which is the same, of the two $(F_1 + F_2)$ and F_3 passes, we have as before

$$\begin{aligned}(F_1 + F_2 + F_3)\, z'' &= (F_1 + F_2)\, z' + F_3 z_3 \\ &= F_1 z_1 + F_2 z_2 + F_3 z_3.\end{aligned}$$

In this manner, introducing successively one force at a time, until all have been taken in, and denoting by $\bar{x}\,\bar{y}\,\bar{z}$ the co-ordinates of the point at which the final resultant acts, we shall at length obtain,

$$(F_1 + F_2 + F_3 + \ldots + F_n)\,\bar{z} = F_1 z_1 + F_2 z_2 + F_3 z_3 + \ldots + F_n z_n,$$
$$\text{or, more concisely, } \Sigma F \,.\, \bar{z} = \Sigma\,(Fz).$$

By similar reasoning we shall obtain

$$\Sigma F \,.\, \bar{y} = \Sigma\,(Fy),$$
$$\text{and } \Sigma F \,.\, \bar{x} = \Sigma\,(Fx).$$

The last three equations determine the values of $\bar{x}\,\bar{y}\,\bar{z}$; and since those values do not contain any terms depending on the inclinations (to the co-ordinate axes) of the lines in which the forces act, those forces may be turned about the points on which they act without affecting the position of the point whose co-ordinates are $\bar{x}\,\bar{y}\,\bar{z}$. On this account this point is called the *centre of parallel forces.*

122. Def. The product of a force into the distance of the point on which it acts from a plane, is called the *moment of the force with respect to the plane.* Hence $\Sigma\,(Fx)$, $\Sigma\,(Fy)$, $\Sigma\,(Fz)$ are the sums of the moments of the forces with respect to the planes of yz, xz, xy: and $\Sigma F \,.\, \bar{x}$, $\Sigma F \,.\, \bar{y}$, $\Sigma F \,.\, \bar{z}$, are the moments of their resultant with respect to the same planes. Hence, remembering that the co-ordinate planes were taken in any position, it follows, that *the sum of the moments of any parallel forces with respect to a plane is equal to the moment of their resultant with respect to the same plane.*

123. If the proposed plane be drawn through the centre of parallel forces, the moment of the resultant with respect to it will be zero; consequently, *the sum of the moments of any parallel forces with respect to any plane passing through their centre is equal to zero.*

124. If ΣF be equal to zero, there is then no centre of parallel forces, as we likewise know from Art. 73.

125. The formulæ of (121) are true if the co-ordinates are oblique: and in that case $\Sigma\,(Fx)$, $\Sigma\,(Fy)$, $\Sigma\,(Fz)$ are called the

oblique moments of the forces with respect to the co-ordinate planes of yz, zx, xy.

THE CENTRE OF GRAVITY.

126. It has been found by experiment, that under the exhausted receiver of an air-pump bodies of unequal magnitudes, and differing altogether in their nature and form (such as a piece of lead, a shilling, a feather, &c.) fall from the top to the bottom of the receiver exactly in the same time: from which it has been inferred, that the Earth exerts an equal force on all equal portions of matter; and that the weight of a body at a given place, measured according to the principles laid down in Arts. 7—10, is proportional to the quantity of matter in the body; that is, if M be the quantity of matter in a body whose weight is W at a given place, then

$$W \propto M.$$

But we have stated in Art. 8, that the weight of a body, measured by a standard spring, is not the same at all places of the Earth's surface; it is in fact (as is shewn in *Dynamics*) proportional to the accelerating force of gravity, at the respective places. This force is generally denoted by g; and hence we have for a given body

$$W \propto g.$$

Consequently, for different bodies at different places $W \propto Mg$. For reasons stated in Dynamics we assume that

$$W = Mg.$$

127. The size or bulk of a body is called its volume and is denoted by V: but it is necessary to explain, both with regard to V and M, that they are expressed in numbers on the following principle. A known body, composed of matter uniformly diffused through all its parts, is taken as a standard to which all others are referred. The volume and mass of this body are called the *units* of volume and of mass. If a body be V times the size, and contain M times the quantity of matter, of the standard body; V and M are taken as the *measures* of the volume and mass of that body. Also, supposing the matter of the second

body to be uniformly diffused through its parts, if a portion of it of the same size as the unit of volume contains ρ times as much matter, ρ is called the density of the body; and it is evident that

$$M = \rho V.$$

128. The direction in which a body descends when let fall is called the *vertical* direction; it may be discovered by suspending a heavy body by a thread, or by drawing a line perpendicular to the surface of still water. A plane at right angles to the vertical is called a *horizontal* plane; and it is evident, since the Earth is spherical, that the horizontal plane changes its position in passing from place to place: but since the distances of the bodies of systems usually treated of in STATICS are exceedingly small compared with the radius of the Earth (4,000 miles, nearly) we may consider the surface of still water as a horizontal plane to a small extent, and consequently the verticals as parallel.

129. Hence it appears, and from Art. 121, that in every body, and in every rigid system of bodies, there is a certain point through which the resultant of the forces which the Earth exerts on the different parts always passes in every position of the body or system. This point is called *the centre of gravity* of the body or system: it is sometimes also called the *centre of mass.*

130. One property of the centre of gravity, particularly worthy of remark, is, that it does not depend at all upon the intensity of the force of gravity. For divide the whole system into very small equal molecules, the quantity of matter in each being m, and their number n, and denote the force exerted upon a unit of matter by g; then the force exerted on each molecule $= mg$. And if $x_1y_1z_1$, $x_2y_2z_2$, ... be the co-ordinates of the molecules, and $\bar{x}\,\bar{y}\,\bar{z}$ those of the centre of gravity, we have, by Art. 121,

$$\bar{x} = \frac{mgx_1 + mgx_2 + mgx_3 + \dots \text{ to } n \text{ terms}}{mg + mg + mg + \dots \text{ to } n \text{ terms}}$$

$$= \frac{x_1 + x_2 + x_3 + \dots\dots}{n}.$$

$$\text{Similarly, } \bar{y} = \frac{y_1 + y_2 + y_3 + \ldots\ldots}{n},$$

$$\text{and } \bar{z} = \frac{z_1 + z_2 + z_3 + \ldots\ldots}{n}.$$

It appears then, that the co-ordinates of the centre of gravity are the means* of the co-ordinates of the equal molecules, and consequently its position is independent of the intensity of gravity. Hence the centre of gravity of any body is a certain point within it, the place of which depends only on the relative disposition of its equal molecules. The investigation of its place is therefore purely geometrical, and may be applied to any body whatever; and for this reason we often speak of the centre of gravity of bodies far removed from the influence of the Earth, and when, in fact, no reference is intended to be made either to the Earth or to gravity; the point alluded to being no other than the one determined from the geometrical principles just laid down, viz.—that its co-ordinates are the respective means of the co-ordinates of all the equal molecules of which the body is composed.

131. When a body is acted on by no other force than gravity, since the resultant of the forces which act on the particles of the body passes through its centre of gravity, if that point be supported the body will be in equilibrium in every position. For instead of the forces themselves, we may substitute their resultant, which will be counteracted by the point of support, and as this will be the case if the body be turned round that point into any position whatsoever, it follows that there will be equilibrium in any position whatever.

132. And since the resultant may be applied at any point in the line of its direction (Art. 21), if the point of support be not in the centre of gravity, but in any point of a vertical passing through it, the body will be in equilibrium. And conversely, if a body be suspended from any point in it, it will not be at rest till the centre of gravity and the point of suspension are situated in the same vertical.

* Hence the centre of gravity of two equal bodies is the middle point between them.

This property may sometimes be employed in finding the centre of gravity in a practical manner. For if the body be successively suspended from two points in it, and the corresponding verticals be drawn upon or through the body, their common point of intersection will be the centre of gravity.

133. It follows at once, from Art. 131, that if all the particles which are situated in a line passing through the centre of gravity be supported, the body will rest in equilibrium on that line in all positions. And the converse is true, viz.—that if a body rest in equilibrium, in all positions, on a fixed line, the centre of gravity must be in that line; for, unless the centre of gravity were in that line, a position might be found in which the vertical through the centre of gravity did not pass through a point of support, and consequently the body would not be in equilibrium in all positions, which is contrary to the hypothesis.

Hence, if we can find two lines on which a body will rest in all positions, the centre of gravity will be in their common point of intersection.

134. Since the resultant of all the forces of gravity, which act on the particles of a body, may be supposed to act at the centre of gravity, and is equal to their sum (Art. 121), we may, in any investigation in which this resultant is required, suppose the whole mass united at the centre of gravity; and hence it becomes important to know the situation of this point in bodies of different figures.

135. It is not always convenient to divide a proposed body into equal molecules, as was done in Art. 130, it therefore becomes necessary, in that case, to use other formulæ for the determination of the centre of gravity.

Let m_1, m_2, m_3,...... be very small masses into which the body may conveniently be supposed to be divided; $x_1 y_1 z_1$, $x_2 y_2 z_2$, $x_3 y_3 z_3$... their co-ordinates.

Then the forces which urge them are gm_1, gm_2, gm_3,...... respectively; and therefore, substituting in Art. 121, we obtain

$$\bar{x} = \frac{gm_1 \cdot x_1 + gm_2 \cdot x_2 + gm_3 \cdot x_3 + \ldots}{gm_1 + gm_2 + gm_3 + \ldots}$$

$$= \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

$$= \frac{\Sigma\,(mx)}{\Sigma m};$$

and, similarly,

$$\bar{y} = \frac{\Sigma\,(my)}{\Sigma m}, \quad \bar{z} = \frac{\Sigma\,(mz)}{\Sigma m}.$$

136. Since, whatever be the position of the plane yz, we always have

$$\bar{x} \cdot \Sigma m = \Sigma\,(mx),$$

it appears that the moment, with respect to any plane, of the whole mass collected at its centre of gravity, is equal to the sum of the moments of all the molecules, with respect to the same plane.

137. If the origin of co-ordinates be in the centre of gravity, then $\Sigma\,(mx) = 0$, $\Sigma\,(my) = 0$, and $\Sigma\,(mz) = 0$; for $\bar{x}$, $\bar{y}$, and $\bar{z}$ are, in that case, each equal to zero.

138. Since the mass of a body of uniform density is measured by the product of its volume into its density (Art. 127); if $\rho_1, \rho_2, \rho_3, \ldots$ be the densities, and $V_1, V_2, V_3, \ldots$ the volumes of the molecules $m_1, m_2, m_3, \ldots$ we shall have

$$m_1 = \rho_1 V_1, \quad m_2 = \rho_2 V_2, \quad m_3 = \rho_3 V_3, \ldots$$

the molecules being so small, that every part of each one may be considered of uniform density. Hence, by substitution in the formulæ of Art. 135, we have

$$\bar{x} = \frac{\rho_1 V_1 \cdot x_1 + \rho_2 V_2 \cdot x_2 + \rho_3 V_3 \cdot x_3 + \ldots}{\rho_1 V_1 + \rho_2 V_2 + \rho_3 V_3 + \ldots}$$

$$= \frac{\Sigma\,(\rho V x)}{\Sigma\,(\rho V)},$$

$$\text{and } \bar{y} = \frac{\Sigma(\rho V y)}{\Sigma(\rho V)},$$

$$\bar{z} = \frac{\Sigma(\rho V z)}{\Sigma(\rho V)}.$$

139. If the density of the whole system be the same in every part, then $\rho_1 = \rho_2 = \rho_3 \ldots$ and these formulæ are simplified by dividing out ρ, thus,

$$\bar{x} = \frac{\Sigma(Vx)}{\Sigma V}, \quad \bar{y} = \frac{\Sigma(Vy)}{\Sigma V}, \quad \bar{z} = \frac{\Sigma(Vz)}{\Sigma V}.$$

But it is to be carefully observed, that these formulæ are only to be applied to such bodies as are of homogeneous materials.

140. The general application of these formulæ depends on the Integral Calculus, but there are a few cases which can be made to depend upon the more simple principles of Art. 133, and with them we shall accordingly commence our series of examples on the subject of finding the position of the centre of gravity in bodies of various forms.

All bodies will be supposed homogeneous, or of uniform density, unless the contrary is mentioned.

141. *If through any figure a plane can be drawn, so that the figure shall be symmetrical with regard to it; that is, so that the two parts of the figure which are situated on opposite sides of that plane are perfectly similar and equal; the centre of gravity is in that plane.*

For the volume of the body being similarly disposed on the two sides of this plane, the moment of the volume on one side is exactly equal to the moment of that on the other side, with respect to that plane, and these moments will have contrary signs, and therefore their sum will be equal to zero. But this sum (Art. 139) is equal to the moment of the whole volume, collected at its centre of gravity, with respect to the same plane; which cannot be the case unless the centre of gravity be in that plane.

142. Hence, if we can find two such planes differently situated, the centre of gravity will be in the line of their intersection; and if we can find a third plane, the centre of gravity will be that point where it cuts the line of intersection of the other two; in other words, it will be the common point of intersection of any three planes, by which the figure can be symmetrically divided.

143. It follows, from these properties,—

(1) That the centre of gravity of a sphere, or of a spheroid, or of a cube, is its centre.

(2) That the centre of gravity of a parallelopiped is the middle point of one of its diagonals; and of a cylinder the middle point of its axis.

(3) That the centre of gravity of any figure of revolution is some point in the axis.

144. When we speak of the centre of gravity of a line, or of a plane figure, it is to be understood that the line consists of material particles, and the plane figure of a single lamina of particles, or else, that the thickness is everywhere the same, and inconsiderable.

145. Hence the centre of gravity of a straight line is its middle point; of a circle, or ellipse, or square, its centre; and it will follow, from reasoning precisely similar to that of Art. 141, that if we can draw two straight lines in a plane, by each of which the figure is divided into two equal and symmetrical parts, the centre of gravity is the point of their intersection. This property will enable us to determine at once, by inspection, the centre of gravity of almost all regular plane figures.

146. *To find the centre of gravity of a plane triangle.*

Let ABC (fig. 26) be the triangle, bisect one of the sides as BC in D, and join AD. Then we may suppose the triangle made up of material particles, arranged in lines parallel to BC; let bc be any one of them. Then, by the similar triangles BAD, bAd,

$$BD : DA :: bd : dA,$$

and, similarly, $DA : DC :: dA : dc$,

$$\therefore\ BD : DC :: bd : dc.$$

But $BD = DC$, therefore $bd = dc$; and consequently, d is the centre of gravity of bc.

Similarly, the centre of gravity of every other line, parallel to BC, of which the triangle consists is somewhere in AD; consequently the whole triangle would rest in equilibrium on AD, and therefore its centre of gravity is in AD (Art. 133). In the same manner it would appear that the centre of gravity of the whole triangle is in BE, which bisects AC, and hence G, the point of intersection of AD and BE, is the point required.

Join DE, then because CA, CB are divided at E, D in the same proportion, viz. each bisected, therefore DE is parallel to AB; and, therefore, the angle DEG is equal to the angle ABG, and angle EDG to the angle BAG, and consequently the triangles ABG, DEG are similar;

$$\therefore\ AG : DG :: AB : DE$$
$$:: AC : EC :: 2 : 1.$$

Hence $AG = 2DG$,

$$\text{and } \therefore\ AD = AG + DG = 3DG;$$
$$\therefore\ DG = \tfrac{1}{3}AD.$$

147. *If three equal bodies have their centres of gravity situated in the three angular points of a triangle, the centre of gravity of these bodies will coincide with that of the triangle.*

Let A, B, C be the centres of gravity of the three equal bodies, then BD being equal to DC, the two bodies at B, C will be in equilibrium on D; and therefore the three bodies will be in equilibrium on a line passing through A, D; in the same manner they will be in equilibrium on BE, and therefore G is their common centre of gravity.

Hence (Art. 130) the distance of the centre of gravity of a triangle from any plane, is the mean of the distances of its angular points from the same plane.

148. *To find the centre of gravity of a quadrilateral figure.*

Let $ABCD$ (fig. 27) be the trapezium; AC, BD its diagonals intersecting in E; G its centre of gravity; draw GI, GK parallel to the diagonals. Then, supposing the trapezium to be made up of the two triangles ADC, ABC, we have (Art. 130),

$$\begin{aligned}&(\text{trapezium } ABCD) \,.\, (\text{perpendicular from } G \text{ on } AC)\\ &- (\Delta ABC) \,.\, (\text{perpendicular from its centre of gravity on } AC)\\ &- (\Delta ABC) \,.\, (\text{perpendicular from its centre of gravity on } AC)\\ &= \tfrac{1}{3} (\Delta ABC) \,.\, (\text{perpendicular from } B \text{ on } AC)\\ &- \tfrac{1}{3} (\Delta ADC) \,.\, (\text{perpendicular from } D \text{ upon } AC).\end{aligned}$$

Now the triangles ABC, ADC, having a common base AC, are proportional to the perpendiculars from B and D on AC, which are also proportional to BE, DE respectively; hence, in the above equation, instead of the triangles ABC, ADC, and the trapezium, which is their sum, write respectively the quantities BE, DE, and $BE+DE$, to which they are proportional; and, instead of the perpendiculars from B, D and G, or I, which is equal to it, write respectively BE, DE, and EI, which are proportional to them; and then we have

$$\begin{aligned}(BE+DE) \,.\, EI &= \tfrac{1}{3} BE^2 - \tfrac{1}{3} DE^2\\ &= \tfrac{1}{3} (BE+DE)(BE-DE);\\ \therefore\ EI &= \tfrac{1}{3} (BD-DE).\end{aligned}$$

$$\text{And, similarly, } EK = \tfrac{1}{3} (AE - CE).$$

Hence, setting off EI equal to one-third of the excess of EB above ED; and EK equal to one-third of the excess of AE above EC; and drawing IG, KG parallel to the diagonals of the trapezium, G will be the point required.

149. To find the centre of gravity of any other rectilinear figure we must divide it into triangles, and suppose each triangle collected at its own centre of gravity; we can then find the common centre of gravity of the whole by the formulæ of Art. 139.

150. *To find the centre of gravity of a triangular pyramid.*

Let A (fig. 28) be the vertex, and BCE the base of the pyramid. E, H the centres of gravity of the base and the face ACD. Join AE, BH, BE, AH. Then, because E is the centre of gravity of the base, therefore BE produced, bisects CD. For a similar reason, AH produced, bisects CD; and therefore BE, AH intersect in F; consequently, AE, BH, which are in the plane ABF, intersect each other in some point G.

Now we may suppose the pyramid made up of triangular laminæ of particles, situated in planes parallel to the base; let cbd be one of them, cutting AF in f, and AE in e. This triangle is, of course, exactly similar to the base of the pyramid, and being parallel to it, cd must be parallel to CD; and therefore the triangles CAF, cAf are similar,

$$\therefore \; cf : Af :: CF : AF;$$

for a similar reason, $\underline{Af : df :: AF : DF;}$

$$\therefore \; cf : df :: CF : DF;$$

but CF being equal to DF, cf must be equal to df, and consequently the centre of gravity of the triangle cbd must be in the line bf. Again, AFB being cut by parallel planes, fb must be parallel to FB, and the triangles FAE, fAe are similar,

$$\therefore fe : Ae :: FE : AE;$$

but, for a similar reason,

$$\underline{Ae : be :: AE : BE,}$$

$$\therefore \; fe : be :: FE : BE.$$

But $BE = 2FE$, and therefore $be = 2fe$, consequently e is the centre of gravity of the lamina bcd. In the same manner it may be proved that all the triangular laminæ of which we have supposed the pyramid to consist have their centres of gravity in AE, wherefore the pyramid would balance on AE in all positions; and, consequently, the centre of gravity is in that line. For like reasons, it is in the line BH, and therefore G, the point of intersection of AE and BH, is the centre of gravity of the pyramid.

Join HE. Then, because $FE : FB :: 1 : 3 :: FH : FA$, therefore HE is parallel to AB, consequently the triangles HEG, BAG are similar;

$$\therefore\ GE : AG :: EH : AB :: FE : FB :: 1 : 3;$$

$$\therefore\ AG = 3GE;$$

$$\therefore\ AE = AG + GE = 4GE;$$

$$\therefore\ GE = \tfrac{1}{4} . AE.$$

Hence, join the vertex and the centre of gravity of the base, and the centre of gravity of the solid will be at the distance of one-fourth of this line from the base.

151. It may be shewn, by a method very similar to the one in Art. 147, that if four equal bodies be placed in the four angular points of the pyramid their common centre of gravity will coincide with the centre of gravity of the pyramid; and that the distance of the centre of gravity of any triangular pyramid, from any plane, is equal to the mean of the distances of its angular points from the same plane.

152. The line joining the centre of gravity of the base BCD, and that of any parallel section bcd of the pyramid being produced, passes through the vertex A.

153. If a plane be drawn through the centre of gravity of the pyramid parallel to the base, a fourth part of any line drawn from the vertex to a point in the base will be intercepted between this plane and the base.

For a fourth part of AE is intercepted, and therefore (Eucl. XI. 16) every line from the vertex to the base is divided in the same proportion.

154. Hence, if a perpendicular be drawn from A upon the base, a fourth part of it will be intercepted between the base and a plane parallel to it through the centre of gravity of the pyramid. And, conversely, if we take a point in the perpendicular at the distance of one-fourth of its length from the base, a plane being drawn through that point parallel to the base will pass through the centre of gravity of the pyramid; consequently, all

other triangular pyramids between the same parallel planes will have their centres of gravity situated in that plane.

155. *To find the centre of gravity of any pyramid.*

Let g (fig. 29) be the centre of gravity of the base of the pyramid; join Ag. Then, by a method exactly similar to the one pursued in Art. 150, it may be shewn that the centres of gravity of all the plane laminæ, parallel to the base, of which the pyramid may be supposed to be made up, are in Ag, and consequently the centre of gravity of the pyramid is in Ag.

But we can divide the base $BCDEF$ into triangles, and suppose the pyramid made up of triangular pyramids, constituted upon these triangles as bases, and having the common vertex A. These, by the last article, will have their centres of gravity in a plane parallel to the base $BCDEF$, which divides Ag in G, so that $Gg = \frac{1}{4}Ag$; consequently the centre of gravity of the whole pyramid will be in that plane, and as it is also in Ag it must be at G.

156. There is nothing in this demonstration to limit the number of sides of the base of the pyramid, and therefore in a cone, upon a curvilinear base of any form whatever, which we may suppose a polygon of an infinite number of sides, the centre of gravity will be found, by joining the vertex and the centre of gravity of the base, and taking a point in that line at the distance of one-fourth of its length from the base.

157. *To find the centre of gravity of the frustum of a cone or pyramid cut off by a plane parallel to the base.*

Let BCD (fig. 30), bcd be the two ends of the frustum, which are, of course, similar figures; g, g' their centres of gravity; G the centre of gravity of the frustum, which will be in the line gg', because the centre of gravity of every lamina parallel to the base is in that line. Now complete the frustum into a pyramid, its vertex A will be in gg' produced (Art. 152); and put a, b for the lengths of corresponding parts of the two ends of the frustum, and c for gg'.

Then Ag' and Ag being like dimensions of the upper pyramid and the whole pyramid, as are also b and a; and, because the like dimensions of similar figures are proportional;

$$\therefore\ a : b :: Ag : Ag';$$

$$\therefore\ a : a-b :: Ag : Ag - Ag' = gg' = c;$$

$$\therefore\ Ag = \frac{ac}{a-b}.$$

$$\text{Similarly, } Ag' = \frac{bc}{a-b}.$$

Now, measuring along gA, the distance of the centre of gravity of the whole pyramid from $g = \frac{1}{4} . \frac{ac}{a-b}$; and the distance of the centre of gravity of the upper pyramid from $g' = \frac{1}{4} . \frac{bc}{a-b}$, and therefore, measuring from g, it $= c + \frac{1}{4} . \frac{bc}{a-b}$; also, putting x for the distance of the centre of gravity of the frustum from g, measuring along gA, we have, by Art. 139,

$$(\text{whole pyramid}) . \tfrac{1}{4} . \frac{ac}{a-b}$$

$$= (\text{frustum}) . x + (\text{upper pyramid}) . \left(c + \tfrac{1}{4} . \frac{bc}{a-b}\right).$$

But similar solid figures are as the cubes of their like dimensions, wherefore the whole pyramid, the upper pyramid, and the frustum, which is the difference between them, are proportional to a^3, b^3, and $a^3 - b^3$ respectively; and substituting these in the last equation for the quantities to which they are proportional, we have

$$a^3 . \tfrac{1}{4} . \frac{ac}{a-b} = (a^3 - b^3)\, x + b^3 . \left(c + \tfrac{1}{4} . \frac{bc}{a-b}\right);$$

$$\therefore\ (a^3 - b^3)\, x = \frac{c}{4}\left\{\frac{a^4 - b^4}{a-b} - 4b^3\right\}$$

$$= \frac{c}{4} . (a^3 + a^2 b + ab^2 + b^3 - 4b^3)$$

$$= \frac{c}{4} . \{(a^3 - b^3) + (a^2 - b^2)\, b + (a - b)\, b^2\};$$

therefore, by dividing the equation by $a - b$,

$$(a^2 + ab + b^2)\, x = \frac{c}{4} \cdot (a^2 + ab + b^2 + ab + b^2 + b^2)$$

$$= \frac{c}{4} \cdot (a^2 + 2ab + 3b^2)\,;$$

$$\therefore\ x = \frac{c}{4} \cdot \frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

GENERAL PROPERTIES OF THE CENTRE OF GRAVITY.

158. *If the mass of each particle of a system be multiplied by the square of its distance from a given point, the sum of the products will be the least possible when the given point is the centre of gravity of the system.*

Let G the centre of gravity of the system be taken for the origin of co-ordinates; and put a, b, c for the co-ordinates of the given point O; $x_1\, y_1\, z_1$, $x_2\, y_2\, z_2 \ldots$ for those of the particles m_1, $m_2 \ldots$ of which the system consists.

Then

$$(Om_1)^2 = (x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2$$

$$= x_1^2 + y_1^2 + z_1^2 + a^2 + b^2 + c^2 - 2ax_1 - 2by_1 - 2cz_1$$

$$= (Gm_1)^2 + (GO)^2 - 2ax_1 - 2by_1 - 2cz_1\,;$$

because $(Gm_1)^2 = x_1^2 + y_1^2 + z_1^2$, and $GO^2 = a^2 + b^2 + c^2$.

Hence $m_1\,(Om_1)^2$

$$= m_1 \cdot (Gm_1)^2 + m_1 \cdot (GO)^2 - 2a \cdot m_1 a - 2b \cdot m_1 y_1 - 2c \cdot m_1 z_1.$$

Similarly, $m_2 \cdot (Om_2)^2$

$$= m_2 \cdot (Gm_2)^2 + m_2 \cdot (GO)^2 - 2a \cdot m_2 x_2 - 2b \cdot m_2 y_2 - 2c \cdot m^2 y_2,$$

$m_3 \cdot (Om_3)^2$

$$= m_3 \cdot (Gm_3)^2 + m_3 \cdot (GO)^2 - 2a \cdot m_3 x_3 - 2b \cdot m_3 y_3 - 2c \cdot m_3 z_3,$$

$$\cdots\cdots\cdots = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and consequently, by adding the corresponding sides of the equations together,

$$\begin{aligned} m_1 . (Om_1)^2 + m_2 . (Om_2)^2 + m_3 . (Om_3)^2 + \ldots\ldots\ldots \\ = m_1 . (Gm_1)^2 + m_2 . (Gm_2)^2 + m_3 . (Gm_3)^2 + \ldots\ldots \\ + (m_1 + m_2 + m_3 + \ldots\ldots) . (GO)^2 \\ - 2a . (m_1x_1 + m_2x_2 + m_3x_3 + \ldots\ldots) \\ - 2b . (m_1y_1 + m_2y_2 + m_3y_3 + \ldots\ldots) \\ - 2c . (m_1z_1 + m_2z_2 + m_3z_3 + \ldots\ldots). \end{aligned}$$

But, because the centre of gravity of the system is in the origin of co-ordinates, we have, by Art. 137,

$$0 = m_1x_1 + m_2x_2 + m_3x_3 + \ldots\ldots$$

$$0 = m_1y_1 + m_2y_2 + m_3y_3 + \ldots\ldots$$

$$0 = m_1z_1 + m_2z_2 + m_3z_3 + \ldots\ldots$$

Consequently,

$$\begin{aligned} m_1 . (Om_1)^2 + m_2 . (Om_2)^2 + m_3 . (Om_3)^2 + \ldots\ldots \\ = m_1 . (Gm_1)^2 + m_2 . (Gm_2)^2 + m_3 . (Gm_3)^2 + \ldots\ldots \\ + (m_1 + m_2 + m_3 + \ldots\ldots) . (GO)^2 \end{aligned}$$

or, $\Sigma \{m\,(Om)^2\} = \Sigma\,\{m\,(Gm)^2\} + \Sigma m . (GO)^2.$

From this equation it appears, that the sum of the products of each particle into the square of its distance from the point O, is greater than $\Sigma\,\{m\,(Gm)^2\}$ by the quantity $\Sigma m . GO^2$; and since $\Sigma\,\{m\,(Gm)^2\}$ does not depend at all upon the position of the point O, the sum will be the least possible when $GO = 0$, that is, when the point O is in the centre of gravity of the system.

159. Cor. 1. So long as the distance of O from G remains the same the quantity $\Sigma\,\{m\,(Gm)^2\} + \Sigma m . GO^2$ retains the same value; if, therefore, O be fixed in space, and the body be made to turn round its centre of gravity, the sum of the products of

each particle of the system into the square of its distance from O remains unaltered.

160. Cor. 2. The last two articles are equally true if $m_1, m_2, m_3 \dots$ be large bodies instead of single particles, observing, in that case, that $x_1\, y_1\, z_1,\, x_2\, y_2\, z_2,\, x_3\, y_3\, z_3 \dots$ will be the coordinates of their respective centres of gravity.

161. Cor. 3. Suppose the bodies each equal to m, and let their number be n, then

$$\begin{aligned}\Sigma\,\{m\,(Om)^2\} &= m_1\,.\,(Om_1)^2 + m_2\,.\,(Om_2)^2 + m_3\,.\,(Om_3)^2 + \dots\dots \\ &= m\,\{(Om_1)^2 + (Om_2)^2 + (Om_3)^2 + \dots\dots\} \\ &= m\,.\,\Sigma\,(Om)^2;\end{aligned}$$

and, similarly, $\Sigma\,\{m\,(Gm)^2\} = m\,.\,\Sigma\,(Gm)^2$; also $\Sigma m = m_1 + m_2 + m_3 + \dots\dots = m + m + m + \dots\dots$ to n terms $= nm$; consequently, by substituting in the equation of Art. 158, we obtain

$$m\,.\,\Sigma\,(Om)^2 = m\,.\,\Sigma\,.\,(Gm)^2 + nm\,.\,(GO)^2;$$

$$\therefore\ \Sigma\,(Om)^2 = \Sigma\,.\,(Gm^2) + n\,.\,(GO)^2.$$

It appears then, that *in a system of* n *equal bodies, the sum of the squares of the distances of their centres of gravity from a given point, is greater than the sum of the squares of the corresponding distances from the centre of gravity of the system, by* n *times the square of the distance of this latter from the given point.*

162. Cor. 4. Hence, if ABC be a triangle, G its centre of gravity, and O a point situated either in the plane of the triangle or not, we have

$$AO^2 + BO^2 + CO^2 = AG^2 + BG^2 + CG^2 + 3\,.\,GO^2.$$

And in a triangular pyramid whose angular points are A, B, C, D, and centre of gravity G,

$$\begin{aligned}&AO^2 + BO^2 + CO^2 + DO^2 \\ &\qquad = AG^2 + BG^2 + CG^2 + DG^2 + 4\,.\,GO^2.\end{aligned}$$

For, by Art. 147, the centre of gravity of the triangle coincides with that of three equal bodies placed at its angular points;

and the centre of gravity of the pyramid with that of four equal bodies at its angular points, (Art. 151).

163. *If each particle of a system be multiplied, as in Art. 158, by the square of its distance from a given point, the sum of the products will be greater than it would be if the whole system were collected at its centre of gravity, by a quantity which is found by multiplying the products of the bodies taken two and two respectively, by the squares of their mutual distances, and dividing the sum of these products by the sum of all the bodies.*

For let O be the given point, G the centre of gravity of the system of particles or bodies m_1, m_2, m_3... Take O for the origin, and let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of G; $x_1 y_1 z_1$, $x_2 y_2 z_2$, $x_3 y_3 z_3$, ... those of m_1, m_2, m_3 ...; also, let $(m_1 m_2)$, $(m_1 m_3)$, $(m_2 m_3)$,... be used to denote the distances between m_1 and m_2, m_1 and m_3, m_2 and m_3 ......

Then, by Art. 135,

$$\bar{x} \,.\, \Sigma m = m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots$$
$$\bar{y} \,.\, \Sigma m = m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots$$
$$\bar{z} \,.\, \Sigma m = m_1 z_1 + m_2 z_2 + m_3 z_3 + \ldots$$

squaring each of these equations and adding the results we obtain

$$\begin{aligned} OG^2 \,.\, (\Sigma m)^2 \,. = m_1^2 \,.\, (Om_1)^2 + m_2^2 \,.\, (Om_2)^2 + m_3^2 \,.\, (Om_3)^2 + \ldots \\ + 2m_1 m_2 \,.\, (x_1 x_2 + y_1 y_2 + z_1 z_2) + \ldots\ldots \\ + 2m_1 m_3 \,.\, (x_1 x_3 + y_1 y_3 + z_1 z_3) + \ldots\ldots \\ + 2m_2 m_3 \,.\, (x_2 x_3 + y_2 y_3 + z_2 z_3) + \ldots\ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

by writing OG^2, $(Om_1)^2$, $(Om_2)^2$, $(Om_3)^2$.... for their equals $\bar{x}^2 + \bar{y}^2 + \bar{z}^2$, $x_1^2 + y_1^2 + z_1^2$, $x_2^2 + y_2^2 + z_2^2$, $x_3^2 + y_3^2 + z_3^2$ ...... respectively.

But $(m_1 m_2)$ being the distance between two points whose co-ordinates are $x_1 y_1 z_1$, $x_2 y_2 z_2$, we have

$$(m_1m_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$
$$= x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2(x_1x_2 + y_1y_2 + z_1z_2)$$
$$= (Om_1)^2 + (Om_2)^2 - 2(x_1x_2 + y_1y_2 + z_1z_2);$$

$$\therefore\ 2m_1m_2(x_1x_2 + y_1y_2 + z_1z_2)$$
$$= m_1m_2\{(Om_1)^2 + (Om_2)^2 - (m_1m_2)^2\}.$$

Similarly,

$$2m_1m_3(x_1x_3 + y_1y_3 + z_1z_3)$$
$$= m_1m_3\{(Om_1)^2 + (Om_3)^2 - (m_1m_3)^2\},$$
$$2m_2m_3(x_2x_3 + y_2y_3 + z_2z_3)$$
$$= m_2m_3\{(Om_2)^2 + (Om_3)^2 - (m_2m_3)^2\}.$$

Consequently, by substitution,

$$OG^2(\Sigma m)^2 = m_1^2 \,.\, (Om_1)^2 + m_2^2 \,.\, (Om_2)^2 + m_3^2 \,.\, (Om_3)^2 + \dots$$
$$+ m_1m_2\{(Om_1)^2 + (Om_2)^2 - (m_1m_2)^2\}$$
$$+ m_1m_3\{(Om_1)^2 + (Om_3)^2 - (m_1m_3)^2\}$$
$$+ m_2m_3\{(Om_2)^2 + (Om_3)^2 - (m_2m_3)^2\}$$
$$+ \dots\dots\dots\dots\dots\dots$$
$$= (m_1 + m_2 + m_3 + \dots\dots)\, m_1 (Om_1)^2$$
$$+ (m_1 + m_2 + m_3 + \dots\dots)\, m_2 (Om_2)^2$$
$$+ (m_1 + m_2 + m_3 + \dots\dots)\, m_3 (Om_3)^2 + \dots\dots$$
$$- m_1m_2 \,.\, (m_1m_2)^2 - m_1m_3 \,.\, (m_1m_3)^2 - m_2m_3 \,.\, (m_2m_3)^2 - \dots$$
$$= \Sigma m \,.\, \Sigma\{m(Om)^2\} - \Sigma\{m_1m_2 \,.\, (m_1m_2)^2\};$$

the term $\Sigma\{m_1m_2 \,.\, (m_1m_2)^2\}$ being understood to represent the sum of the products of the particles, taken two and two, into the squares of their mutual distances.

Hence dividing by Σm and transposing,

$$\Sigma\{m(Om)^2\} = (\Sigma m) \,.\, OG^2 + \frac{\Sigma\{m_1m_2 \,.\, (m_1m_2)^2\}}{\Sigma m},$$

which expresses the property to be proved.

164. Cor. 1. From Art. 158, we have

$$\Sigma \{m\,(Om)^2\} = \Sigma \{m\,(Gm)^2\} + \Sigma m\,.\,(GO)^2;$$

which, substituted in the equation above obtained, gives

$$\Sigma \{m\,(Gm)^2\} = \frac{\Sigma \{m_1 m_2\,.\,(m_1 m_2)^2\}}{\Sigma m}.$$

A result which might have been obtained at once without the aid of Art. 158 by supposing O to coincide with G.

165. Cor. 2. If now, as in Art. 161, we suppose all the bodies equal and n in number, the last equation becomes

$$m\,.\,\Sigma (Gm)^2 = \frac{m}{n}\,.\,\Sigma\,(m_1 m_2)^2;$$

$$\therefore\ \Sigma\,(m_1 m_2) = n\,.\,\Sigma\,(Gm)^2.$$

Hence, *in any system of* n *equal bodies, the sum of the squares of the lines joining their centres of gravity, two and two, is equal to* n *times the sum of the squares of the distances of those points from the centre of gravity of the system.*

166. Cor. 3. Consequently, in the case of the triangle (Art. 147),

$$BC^2 + AC^2 + AB^2 = 3\,.\,(AG^2 + BG^2 + CG^2).$$

Hence *the sum of the squares of the three sides of a triangle is equal to three times the sum of the squares of the distances of its angular points from its centre of gravity.*

167. Cor. 4. In the case of the triangular pyramid we have

$$AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2$$
$$= 4\,(AG^2 + BG^2 + CG^2 + DG^2).$$

Hence *the sum of the squares of the six edges of a triangular pyramid is equal to four times the sum of the squares of the distances of its angular points from its centre of gravity.*

168. *When a system of bodies is in equilibrium under the action of gravity only, the altitude of the centre of gravity of the system is in general a maximum or a minimum.*

Let $m_1, m_2, m_3 \ldots$ be the particles of the system in equilibrium: $z_1, z_2, z_3 \ldots$ their respective altitudes above a fixed horizontal plane; $\bar{z}$ the altitude of the centre of gravity of the system above the same plane; g the accelerating force of gravity; then $m_1 g, m_2 g, m_3 g \ldots$ are the forces acting upon the particles of the system. Let now the system be disturbed in a manner subject to the same restrictions as were pointed out in the Chapter on virtual velocities, (i. e. rods must not be bent, cords must be kept of invariable length, contacts must not be broken, &c.) and let $dz_1, dz_2, dz_3 \ldots$ be the virtual velocities of the respective particles, then by Art. 113, because there is equilibrium,

$$m_1 g \, . \, dz_1 + m_2 g \, . \, dz_2 + m_3 g \, . \, dz_3 + \ldots = 0,$$

$$\text{or } \Sigma \, (mdz) = 0.$$

$$\text{But since } \Sigma m \, . \, \bar{z} = \Sigma \, (mz)\,;$$

$$\therefore \; \Sigma m \, . \, d\bar{z} = \Sigma \, (mdz) = 0, \quad \therefore \; d\bar{z} = 0.$$

Now $d\bar{z}$ is the differential of $\bar{z}$, or second term of Taylor's Theorem, and this being equal of zero, it follows that $\bar{z}$ is in general a maximum or minimum.

It has been stated that the principle of virtual velocities extends only of quantities to the first order of smallness, that is, to the second term of Taylor's Theorem only; the equilibrium of the system therefore does not require that $\Sigma \, (md^2 z)$ shall be equal to zero, though it may happen to be so in particular cases; and the algebraic sign of $d^2\bar{z}$ will decide whether $\bar{z}$ is a maximum or a minimum.

169. COR. Since the centre of gravity of the system is the point through which the resultant $\Sigma \, (mg)$, or $g\Sigma m$ of all the forces $m_1 g, m_2 g \ldots$ passes, and seeing that this resultant acts in a downward direction, it appears that, if the system be disturbed, the tendency of gravity is to make the centre of gravity descend: but if the geometrical constitution of the system be such that in passing out of a position of equilibrium the centre of gravity can only ascend, the ascent will be *opposed* by gravity; that is, in this case gravity tends to bring the system back again into its position of equilibrium. But if the constitution of the system be

such that in passing out of equilibrium the centre of gravity cannot but descend, it is assisted in its descent by gravity, and there is no tendency to return towards the position from which it set out. Hence it follows:

(1) That if the altitude of the centre of gravity be a minimum, the system when disturbed will return by the action of gravity towards the position from which it was disturbed. This is therefore called a position of *stable equilibrium.*

(2) That if the altitude of the centre of gravity be a maximum, the system when disturbed will recede by the action of gravity still farther from the position of equilibrium. This is therefore called a position of *unstable equilibrium.*

(3) That if the centre of gravity neither ascend nor descend when the system is disturbed, it still continues in a position of equilibrium. This is therefore called a position of *neuter equilibrium.*

170. *If a body be placed with its base upon a plane it will stand or fall according as a vertical through its centre of gravity falls within or without its base.*

Let AB (figs, 31, 32) be the base of the body, G its centre of gravity; draw a vertical through G meeting the plane on which the body is placed in H; H falling within the base in fig. 31, and without it in fig 32.

Every particle of the body is acted on by the force of gravity, and we have shewn that the centre of gravity is the point at which the resultant of the forces may be supposed to act: this resultant is equal to their sum, that is, it is equal to (W) the whole weight of the body. We may therefore suppose the body to be without weight, and that a force acts at G equal to W. In fig. 31, we may suppose this force to be transmitted to H, which being in contact with a fixed point of the plane cannot be moved, and therefore W is counteracted, its effects being to make the body stand firm upon its base. But in fig. 32, W cannot be transmitted to a point which is in contact with the plane, and therefore as there is nothing to oppose its action, the point G

will descend, thereby causing the body to turn about the point A.

171. This reasoning applies if the plane on which the body is placed be not horizontal, provided the body be prevented from sliding by the roughness of the plane, or any equivalent cause.

172. If a body be placed on points, instead of a flat base, it will stand or fall according as a vertical through its centre of gravity falls within or without the polygon formed by passing a thread round the points.

173. If there be any case not here considered, it may be disposed of on the following principle. The whole weight of the body may be supposed to act at its centre of gravity; and as it acts in a downward direction, its tendency is to cause that point to *descend*. If the geometrical arrangement of the system be such that it is impossible for it to move so as to permit the centre of gravity to *descend*, it will remain stationary; for in this case the tendency which gravity produces is prevented from taking effect from the construction of the machine.

APPLICATION OF THE INTEGRAL CALCULUS TO FIND THE CENTRE OF GRAVITY OF BODIES.

174. *To find the centre of gravity of a plane curve line.*

Let AB (fig. 33) be the curve line, referred to the rectangular axes Ox, Oy. P any point in AB, and Q very near to P. $x = OM$, $\delta x = MN$, $y = MP$, $s = AP$, $\delta s = PQ$, $u =$ the moment of the arc AP, and $\delta u =$ that of PQ, about Oy.

The moment of PQ about Oy is greater than it would be if PQ were all collected in a point at P;

$$\therefore\ \delta u > x\delta s;$$

and it is less than if PQ were all collected at Q;

$$\therefore\ \delta u < (x + \delta x)\,\delta s.$$

Hence $\frac{\delta u}{\delta s}$ always lies between x and $x + \delta x$, consequently

$$\text{the limit of } \frac{\delta u}{\delta s} = x;$$

but by the principles of the Differential Calculus

$$\text{the limit of } \frac{\delta u}{\delta s} = \frac{du}{ds};$$

$$\therefore \frac{du}{ds} = x;$$

$$\therefore u = \int x ds,$$

the integral to be taken from $x = OC$ to $x = OD$.

But if $\bar{x}\bar{y}$ be the co-ordinates of the centre of gravity of AP, we have by Art. 139,

$$\bar{x}s = u = \int x ds;$$

$$\therefore \bar{x} = \frac{\int x ds}{s}.$$

$$\text{Similarly, } \bar{y} = \frac{\int y ds}{s}.$$

175. These formulæ will suffice for the determination of the point required in any given example: but it may be remarked with respect to these, and other formulæ, which will be investigated for finding the centres of gravity of areas and volumes, that they are not always of convenient application. It is, generally speaking, more easy to work out an example by taking an element δm of the figure, and then applying the equations

$$\bar{x} = \frac{\Sigma(x\delta m)}{\Sigma(\delta m)}, \quad \bar{y} = \frac{\Sigma(y\delta m)}{\Sigma(\delta m)}.$$

If this method be applied to the case investigated in the last Article, we have $\delta m = \delta s$; $\therefore \Sigma(\delta m) = \Sigma\delta s = \int ds = s$; and $\Sigma(x\delta m) = \Sigma(x\delta s) = \int x\delta s)$; and $\therefore \bar{x} = \frac{\int x ds}{s}$, the same result as before

Ex. 1. *To find the centre of gravity of the arc of a semi-cycloid.*

Let BC (fig. 34) be the base, AB the axis, and AC the arc of the semi-cycloid; $x = AM$, $y = MP$, $s = AP$, $2a = AB$; then the equation of the cycloid is

$$y = (2ax - x^2)^{\frac{1}{2}} + a \operatorname{vers}^{-1} \frac{x}{a};$$

$$\therefore\ dy = \left(\frac{2a}{x} - 1\right)^{\frac{1}{2}} dx,$$

$$\text{and } ds = \left(\frac{2a}{x}\right)^{\frac{1}{2}} dx;$$

$$\therefore\ s = 2\sqrt{2ax}.$$

$$\text{Also, } xds = \sqrt{2ax} \,.\, dx;$$

$$\therefore\ \int xds = \frac{2}{3} x \sqrt{2ax};$$

$$\therefore\ \bar{x} = \frac{x}{3} \ldots\ldots\ldots\ldots\ldots (1).$$

Again, $\int yds = ys - \int sdy$

$$= ys - \int 2\sqrt{2ax} \left(\frac{2a}{x} - 1\right)^{\frac{1}{2}} dx$$

$$= ys - 2\sqrt{2a} \int (2a - x)^{\frac{1}{2}} dx$$

$$= ys + \frac{4}{3}\sqrt{2a}\,(2a - x)^{\frac{3}{2}} + C.$$

Now this integral ought to vanish when $x = 0$;

$$\therefore\ C = -\frac{16}{3} a^2;$$

$$\text{and } \int yds = ys + \frac{4}{3}\sqrt{2a}\,(2a - x)^{\frac{3}{2}} - \frac{16}{3} a^2;$$

$$\therefore\ \bar{y} = y + \frac{2}{3} \frac{(2a - x)^{\frac{3}{2}}}{\sqrt{x}} - \frac{8a^2}{3\sqrt{2ax}} \ldots\ldots\ldots\ldots\ldots (2).$$

The equations (1), (2) give the co-ordinates of the centre of gravity of any arc AP: and if we write in them $2a$ for x, we find

$$\frac{2a}{3} \text{ and } \left(\pi - \frac{4}{3}\right) a,$$

for the co-ordinates of the centre of gravity of the arc AC.

Ex. 2. *To find the centre of gravity of an arc of a circle.*

Let AB (fig. 35) be the given arc, O its centre, C its middle point; join OA, OB, OC: and let PQ be a very small element of the arc. Draw Oy perpendicular to OC. $a = OA$, $\alpha = AOC$, $\theta = COP$, $\delta\theta = POQ$: the centre of gravity of AB is manifestly in the line OC, let $\bar{x}$ be its distance from O measured along OC. Then

$$\text{the element } PQ = a\delta\theta,$$

$$\text{its moment about } Oy = a\delta\theta \,.\, a\cos\theta;$$

$$\therefore \text{ moment of the arc } AB = a^2 \int \cos\theta d\theta \text{ from } \theta = -\alpha \text{ to } \theta = +\alpha$$

$$= a^2 \sin\theta, \text{ from } \theta = -\alpha \text{ to } \theta = +\alpha,$$

$$= 2a^2 \sin\alpha,$$

$$\text{and arc } AB = 2a\alpha;$$

$$\therefore \bar{x} = \frac{\Sigma(x\delta m)}{\Sigma(\delta m)}$$

$$= \frac{2a^2 \sin\alpha}{2a\alpha}$$

$$= a\frac{\sin\alpha}{\alpha}.$$

Ex. 3. The equation of a catenary being

$$\frac{x}{a} = \tfrac{1}{2}\left(e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right),$$

and $\bar{x}\bar{y}$ being the co-ordinates of the semi-arc (s), shew that

$$\bar{x} = \frac{x}{2} + \frac{ay}{2s}, \quad \bar{y} = y - \frac{ax}{s}.$$

Ex. 4. The equation of a parabola being $y^2 = 4mx$, shew that the distance of the centre of gravity of the arc, cut off by the latus rectum, from its vertex, is

$$\frac{m}{4} \cdot \frac{3\sqrt{2} - \log_e(1+\sqrt{2})}{\sqrt{2} + \log_e(1+\sqrt{2})}.$$

176. *To find the centre of gravity of a plane area.*

Let $ACDB$ (fig. 33) be the area: then using the same notation as in Art. 174,

$$\text{an element of area} = PN = y\delta x,$$

$$\text{its moment about } Oy = x \,.\, ydx \text{ ultimately};$$

$$\therefore \text{ moment of the whole area about } Oy = \Sigma\,(xy\delta x);$$

$$= \int xydx,$$

$$\text{and the whole area} = \int ydx;$$

$$\therefore \bar{x} = \frac{\int xydx}{\int ydx}.$$

The integrals are to be taken from $x = OC$ to $x = OD$.

Again,

$$\text{the moment of the element } y\delta x \text{ about } Ox$$

$$= ydx \,.\, \frac{y}{2} \text{ ultimately};$$

$$\therefore \text{ moment of the whole about } Ox = \tfrac{1}{2}\int y^2 dx;$$

$$\therefore \bar{y} = \tfrac{1}{2}\frac{\int y^2 dx}{\int ydx}.$$

Ex. 1. *To find the centre of gravity of the area of a semi-parabola.*

Let A (fig. 34) be the vertex, and AB the axis of the parabola; and let $y^2 = 4mx$ be the equation of the curve, where $x = AM$, and $y = MP$; put $AB = a$;

$$\therefore \int xydx = \int \sqrt{4m} \,.\, x^{\frac{3}{2}} dx$$

$$= \frac{2}{5}\sqrt{4m} \,.\, x^{\frac{5}{2}} + C$$

$$= \frac{2}{5}\sqrt{4m} \,.\, a^{\frac{5}{2}} \text{ between } x = 0 \text{ and } x = a.$$

Also $\int y^2 dx = \int 4mx dx$

$$= 2ma^2 \text{ between the same limits:}$$

and $\int y dx = \int \sqrt{4m}\, x^{\frac{1}{2}} dx$

$$= \frac{2}{3}\sqrt{4m}\, a^{\frac{3}{2}} \text{ between the same limits;}$$

$$\therefore\ \bar{x} = \frac{\frac{2}{5}\sqrt{4m}\, a^{\frac{5}{2}}}{\frac{2}{3}\sqrt{4m}\, a^{\frac{3}{2}}} = \frac{3}{5}a = \frac{3}{5}AB;$$

$$\text{and } \bar{y} = \frac{1}{2}\frac{2ma^2}{\frac{2}{3}\sqrt{4m}\, a^{\frac{3}{2}}} = \frac{3}{4}\sqrt{ma} = \frac{3}{8}BC.$$

Ex. 2. *To find the centre of gravity of the area of a circular sector.*

Let AOB (fig. 35) be the given area, x the distance of its centre of gravity from O; then using the same notation as in Ex. 2 of Art. 175, we have

$$\text{elementary area} = \Delta POQ = \tfrac{1}{2}a^2\delta\theta \text{ ultimately;}$$

$$\therefore \text{ area of the sector} = \int \tfrac{1}{2}a^2 d\theta \text{ from } \theta = -\alpha \text{ to } \theta = +\alpha$$
$$= a^2\alpha.$$

Now the elementary area POQ being ultimately a triangle, we may suppose its centre of gravity to be at g, such that $Og = \frac{2}{3}OP = \frac{2}{3}a$: and as the distance of g from $Oy = \frac{2}{3}a\cos\theta$ ultimately, we have the moment of the elementary area QOP about Oy

$$= \frac{1}{2}a^2\delta\theta \,.\, \frac{2}{3}a\cos\theta;$$

$\therefore$ moment of the sector about Oy

$$= \frac{1}{3}a^3 \int \cos\theta d\theta$$

$$= \frac{1}{3}a^3 \sin\theta + C$$

$$= \frac{2}{3}a^3 \sin\alpha \text{ from } \theta = -\alpha \text{ to } \theta = +\alpha;$$

$$\therefore\ \bar{x} = \frac{\frac{2}{3}a^3 \sin\alpha}{a^2\alpha}$$

$$= \frac{2a \sin\alpha}{3\alpha}.$$

Ex. 3. If $\bar{x}\bar{y}$ be the co-ordinates of the centre of gravity of the area of a semi-cycloid whose equation is

$$y = (2ax - x^2)^{\frac{1}{2}} + a \operatorname{vers}^{-1}\frac{x}{a},$$

$$\bar{x} = \frac{7a}{6} \text{ and } \bar{y} = \frac{a\pi}{2}\left(1 - \frac{16}{9\pi^2}\right).$$

Ex. 4. If $\bar{x}\bar{y}$ be the co-ordinates of the centre of gravity of the area cut off from a parabola ($y^2 = 4mx$) by a focal chord inclined to the axis at an angle α,

$$\bar{x} = \frac{m}{5}(3 + 8\cot^2\alpha) \text{ and } \bar{y} = 2m\cot\alpha.$$

Ex. 5. To find the centre of gravity of the area of the quadrant of a circle, whose equation is $x^2 + y^2 = a^2$

$$\bar{x} = \frac{4a}{3\pi} = \bar{y}.$$

Ex. 6. To find the centre of gravity of the node of the lemniscate, whose equation is $r^2 = a^2 \cos 2\theta$,

$$\bar{x} = \frac{\pi a}{4\sqrt{2}}, \quad \bar{y} = 0.$$

177. *To find the centre of gravity of a solid of revolution.*

Let AB (fig. 33) be the curve by the revolution of which round Ox the given solid is generated. Make the same construction and notation as before. Let V denote the volume of the solid generated by the revolution of AMP, and δV that generated by $PMNQ$; $u =$ the moment of V round Oy, and δu that of δV about Oy.

The moment of δV about Oy is greater than it would be if δV were all collected in the circular plane generated by PM, that is,

$$\delta u > x \,.\, \delta V;$$

and it is less than it would be if δV were all collected in the circular plane generated by QN, that is,

$$\delta u < (x + \delta x) \,.\, \delta V.$$

Hence $\frac{\delta u}{\delta x}$ always lies between $x \,.\, \frac{\delta V}{\delta x}$ and $x \,.\, \frac{\delta V}{\delta x} + \delta V.$

Whence, as in Art. 174,

$$\frac{du}{dx} = x \,.\, \frac{dV}{dx};$$

$$\therefore\ u = \int (x dV).$$

But $\bar{x}$, $\bar{y}$ being the co-ordinates of the centre of gravity of V,

$$\bar{x} \,.\, V = u = \int (x dV).$$

Now $dV = \pi y^2 dx$, by the Differential Calculus; and, therefore, $V = \pi \int y^2 dx$; consequently

$$\bar{x} \int y^2 dx = \int xy^2 dx;$$

$$\therefore\ \bar{x} = \frac{\int xy^2 dx}{\int y^2 dx}.$$

From Art. 143, it is manifest that

$$\bar{y} = 0.$$

Ex. 1. *To find the centre of gravity of a hemisphere.*

A hemisphere is generated by the revolution of a quadrant whose equation is

$$y^2 = 2ax^2 - x;$$

$$\therefore\ \int y^2 dx = ax^2 - \frac{1}{3} x^3,$$

which gives, for the whole hemisphere, by writing a for x, the quantity $\frac{2}{3} a^3$.

Again, $$\int xy^2dx = \int (2ax^2 - x^3)\,dx,$$

$$= \frac{2}{3}ax^3 - \frac{1}{4}x^4;$$

which, by writing a for x, becomes $\frac{5}{12}a^4$;

$$\therefore\ \bar{x} = \frac{\frac{5}{12}a^4}{\frac{2}{3}a^3},$$

$$= \frac{5}{8}a = \frac{5}{8} \text{ of the radius.}$$

Ex. 2. Given the altitude (c) and the radii (a, b) of the ends of a parabolic frustum, to find its centre of gravity;

$$\bar{x} = \frac{c}{3} \cdot \frac{a^2 + 2b^2}{a^2 + b^2}, \quad \text{and } \bar{y} = 0:$$

$\bar{x}$ being measured along the axis from the smaller end whose radius is a.

Ex. 3. In a cone, generated by the revolution of a right-angled triangle about one of its sides,

$$\bar{x} = \tfrac{3}{4} \text{ of that side.}$$

Ex. 4. In the solid formed by the revolution of a semi-cycloid about its axis,

$$\bar{x} = \frac{a}{6} \cdot \frac{63\pi^2 - 64}{9\pi^2 - 16}.$$

$\bar{x}$ being measured from the base along the axis.

Ex. 5. In the paraboloid, formed by the revolution of the parabola, whose equation is $y^{m+n} = a^m x^n$,

$$\bar{x} = \frac{m + 3n}{m + 2n} \cdot \frac{x}{2}.$$

178. *To find the centre of gravity of a solid of any form.*

Let Ox, Oy, Oz (fig. 36) be the rectangular co-ordinate axes to which the solid is referred by its equation. Let $ABPC$ be a portion of the surface of the solid, comprehended between the co-ordinate planes xOz, yOz, and the planes $PpNC$, $PpMB$ respectively parallel to them. Through the point S very near to P draw planes $Ssnc$, $Ssmb$ parallel to the former. Let xyz be the co-ordinates of P, and $x+\delta x$, $y+\delta y$, $z+\delta z$ those of S. Then, denoting the volume of the parallelopiped Ps by A, its moment about the axis Ox is greater than if it were all collected in the plane Pq, and less than if collected in the plane Rs; that is, the moment of A is

$$\text{greater than } yA,$$

$$\text{and less than } (y+\delta y)\,A.$$

But now if u be the moment of the solid PO about Ox, the moment of $SBmPn$ about Ox will be (by Taylor's theorem applied to two variables x, y)

$$\begin{aligned} &d_x u\,.\,\delta x + \tfrac{1}{2} d_x^{\,2} u\,.\,(\delta x)^2 + \dots \\ &d_y u\,.\,\delta y + d_x d_y u\,.\,\delta x \delta y + \dots \\ &\qquad\qquad + \tfrac{1}{2} d_y^{\,2} u\,.\,(\delta y)^2 + \dots \\ &\qquad\qquad\qquad\qquad + \dots \end{aligned}$$

and by the same theorem, applied to the variable x, the moment of the solid BmP about Ox is

$$d_x u\,.\,\delta x + \tfrac{1}{2} d_x^{\,2} u\,.\,(\delta x)^2 + \dots$$

and, similarly, the moment of the solid CnP, is

$$d_y u\,.\,\delta y + \tfrac{1}{2} d_y^{\,2} u\,.\,(\delta y)^2 + \dots$$

Subtracting both these from the former, we find the moment of the parallelopiped Ps to be equal to $d_x d_y u\,.\,\delta x \delta y + \dots$; consequently, this quantity always lies between yA and $(y+\delta y)\,A$; and, therefore, $d_x d_y u + \dots$ always lies between

$$y\,.\,\frac{A}{\delta x \delta y} \text{ and } y\,.\,\frac{A}{\delta x \delta y} + \delta y\,.\,\frac{A}{\delta x \delta y}.$$

Now $\dfrac{A}{\delta x \delta y}$ tends to z as its limit, and consequently the two quantities $y\,.\,\dfrac{A}{\delta x \delta y}$ and $y\,.\,\dfrac{A}{\delta x \delta y} + \delta y\,.\,\dfrac{A}{\delta x \delta y}$ tend to equality with yz; and $d_x d_y u + \dots$ which always lies between them, tends to

$d_x d_y u$ as its limit; the three limits are therefore equal; consequently,

$$d_x d_y u = yz\,;$$

$$\therefore\ u = \int_x \int_y (yz).$$

Now the volume of PO is equal to $\int_x \int_y z$, and its moment about Ox is

$$\bar{y} \cdot \int_x \int_y z\,;$$

wherefore, by Art. 139,

$$\bar{y} \cdot \int_x \int_y z = \int_x \int_y (yz) \ldots\ldots\ldots\ldots (2).$$

By a similar investigation, we should find

$$\bar{x} \cdot \int_x \int_y z = \int_x \int_y (xz) \ldots\ldots\ldots\ldots (1).$$

And observing that the centre of gravity of the parallelopiped A is ultimately in its middle point, we should find

$$\bar{z} \cdot \int_x \int_y z = \tfrac{1}{2} \int_x \int_y (z^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Remark. It is evident, that by taking an elementary parallelopiped, at right angles to the plane xOz, we might also obtain

$$\bar{x} \cdot \int_x \int_z y = \int_x \int_z (xy),$$

$$\bar{y} \cdot \int_x \int_z y = \tfrac{1}{2} \int_x \int_z (y^2),$$

$$\bar{z} \cdot \int_x \int_z y = \int_x \int_z (zy)\,;$$

and if the elementary parallelopiped were at right angles to the plane yOz, we should find

$$\bar{x} \cdot \int_y \int_z x = \tfrac{1}{2} \int_y \int_z (x^2),$$

$$\bar{y} \cdot \int_y \int_z x = \int_y \int_z (xy),$$

$$\bar{z} \cdot \int_y \int_z x = \int_y \int_z (xz).$$

These formulæ are in fact, often more convenient than those first given; and which are the most convenient in a given example is to be determined by the form of the body and its situation with respect to the co-ordinate planes; the choice must, however, be left to the skill of the reader, as no general rule can be laid down. In every case, the greatest care is requisite to take the integrals between proper limits.

All the three sets of formulæ are comprehended in the following:—

$$\bar{x} \cdot \int_x \int_y \int_z 1 = \int_x \int_y \int_z x,$$
$$\bar{y} \cdot \int_x \int_y \int_z 1 = \int_x \int_y \int_z y,$$
$$\bar{z} \cdot \int_x \int_y \int_z 1 = \int_x \int_y \int_z z,$$

which may be readily investigated after the manner of Art. 175.

Ex. 1. *To find the centre of gravity of the eighth part of an ellipsoid.*

The equation of the surface of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$\therefore \int_y z = c \int_y \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$$

$$= \frac{c}{b} \cdot \int_y \left\{\left(b^2 - \frac{b^2x^2}{a^2}\right) - y^2\right\}^{\frac{1}{2}}$$

$$= \tfrac{1}{2} \cdot \frac{c}{b} \cdot y \left(b^2 - \frac{b^2x^2}{a^2} - y^2\right)^{\frac{1}{2}}$$

$$+ \tfrac{1}{2} \cdot \frac{c}{b} \cdot \left(b^2 - \frac{b^2x^2}{a^2}\right) \sin^{-1} \cdot \frac{y}{\left(b^2 - \dfrac{b^2x^2}{a^2}\right)^{\frac{1}{2}}} + C$$

$$= \tfrac{1}{2} cy \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} + \frac{bc}{2a^2} (a^2 - x^2) \sin^{-1} \frac{ay}{b\sqrt{a^2 - x^2}} + C.$$

This integral is to be taken from $y = 0$, to that value of y which makes $z = 0$; or from $y = 0$, to $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$; and therefore

$$\int_y z = \frac{bc\pi}{4a^2} (a^2 - x^2);$$

$$\therefore \int_x \int_y z = \frac{bc\pi}{4a^2} \int_x (a^2 - x^2)$$

$$= \frac{bc\pi}{4a^2} \cdot (a^2 x - \tfrac{1}{3} x^3 + C).$$

This integral is to be taken from $x = 0$, to $x = a$; and therefore

$$\int_x \int_y z = \frac{bc\pi}{4a^2} \cdot \frac{2a^3}{3} = \frac{\pi}{6} \cdot abc.$$

Again, to find the value of $\int_x \int_y (xz)$ we observe that

$$\begin{aligned} \int_y (xz) &= x \int_y z \\ &= x \cdot \frac{bc\pi}{4a^2} (a^2 - x^2) \\ &= \frac{bc\pi}{4a^2} (a^2 x - x^3); \end{aligned}$$

$$\begin{aligned} \therefore \int_x \int_y (xz) &= \frac{bc\pi}{4a^2} \cdot \int_x (a^2 x - x^3) \\ &= \frac{bc\pi}{4a^2} (\tfrac{1}{2} a^2 x^2 - \tfrac{1}{4} x^4) + C; \end{aligned}$$

which, taken between the same limits as before, viz. $x = 0$, and $x = a$, gives

$$\int_x \int_y (xz) = \frac{\pi a^2 bc}{16}.$$

$$\text{Hence } \bar{x} \cdot \frac{\pi abc}{6} = \frac{\pi a^2 bc}{16};$$

$$\therefore \bar{x} = \frac{3}{8} a.$$

$$\text{Similarly, } \bar{y} = \frac{3}{8} b;$$

$$\text{and } \bar{z} = \frac{3}{8} c.$$

Ex. 2. To find the centre of gravity of a portion of a paraboloid, comprehended between two planes passing through its axis at right angles to each other.

If a be its length, and b the radius of its base, the co-ordinates of its centre of gravity will be

$$\bar{x} = \frac{2}{3} a, \quad \bar{y} = \bar{z} = \frac{16b}{15\pi}.$$

179. *To find the centre of gravity of a surface of revolution.*

Employing the notation and figure of Art. 177, let u be the moment of the surface generated by the arc AP, and therefore δu the moment of that generated by PQ; let S denote the former, and δS the latter of these surfaces so generated. Then the moment of δS about Oy is greater than if it were all collected in the circumference of the circle described by P, and less than if collected in the circumference of that described by Q, that is,

δu is greater than $x \,.\, \delta S$, and less than $(x + \delta x) \,.\, \delta S$;

$$\therefore \frac{\delta u}{\delta x} \text{ lies between } x\frac{\delta S}{\delta x} \text{ and } x\frac{\delta S}{\delta x} + \delta S.$$

Equating the limits, as before, we have

$$\frac{du}{dx} = x\frac{dS}{dx} = x\, 2\pi y \frac{ds}{dx};$$

$$\therefore u = 2\pi \int (xy\, ds).$$

But

$$u = \text{the moment of } S \text{ about } Oy = \bar{x} S = \bar{x} \,.\, 2\pi \int (y ds);$$

$$\therefore \bar{x} \,.\, 2\pi \int (y ds) = 2\pi \int (xy\, ds);$$

$$\therefore \bar{x} \int (y ds) = \int (xy\, ds).$$

And it is evident, from the symmetrical form of the surface, that $\bar{y} = 0$.

Ex. 1. *To find the centre of gravity of the surface of a cone.*

If a be the altitude and b the radius of the base of the cone, the equation of the line by which the surface is generated is

$$y = \frac{bx}{a};$$

$$\therefore ds = \left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}} dx;$$

$$\therefore \int (y ds) = \int \frac{bx}{a}\left(1+\frac{b^2}{a^2}\right)^{\frac{1}{2}} dx$$

$$= \frac{bx^2}{2a} \cdot \left(1+\frac{b^2}{a^2}\right)^{\frac{1}{2}} + C;$$

which, taken between the limits $x = 0$, and $x = a$, gives

$$\int (y ds) = \tfrac{1}{2} b \sqrt{a^2+b^2}.$$

Also,

$$\int (xy\, ds) = \int \frac{bx^2}{a}\left(1+\frac{b^2}{a^2}\right)^{\frac{1}{2}} dx$$

$$= \frac{bx^3}{3a}\left(1+\frac{b^2}{a^2}\right)^{\frac{1}{2}} + C;$$

which, between the same limits, gives

$$\int (xy\, ds) = \tfrac{1}{3} ab \sqrt{a^2+b^2};$$

$$\therefore \ \bar{x} \,.\, \tfrac{1}{2} b \sqrt{a^2+b^2} = \tfrac{1}{3} ab \sqrt{a^2+b^2};$$

$$\therefore \ \bar{x} = \tfrac{2}{3} a.$$

Ex. 2. To find the centre of gravity of the surface generated by the revolution of an arc of a circle about a diameter.

The centre of gravity bisects the axis of the zone.

Ex. 3. To find the centre of gravity of the surface generated by the revolution of a semi-cycloid about its axis,

$$\bar{x} = \frac{2a}{3} \cdot \frac{\pi - \frac{8}{15}}{\pi - \frac{4}{3}}.$$

Ex. 4. To find the centre of gravity of the surface of a paraboloid.

Taking the focus as origin of polar co-ordinates, we find the distance of the centre of gravity from the directrix

$$= \frac{3m}{5} \cdot \frac{\sec^5 \frac{\theta}{2} - 1}{\sec^3 \frac{\theta}{2} - 1}.$$

Ex. 5. To find the centre of gravity of the surface generated by the revolution of a node of the Lemniscate about its axis.

$$\bar{x} = \frac{a}{6} \cdot \frac{1 - \cos^{\frac{3}{2}} 2\theta}{1 - \cos\theta} = \frac{a}{12} \cdot \frac{2\sqrt{2} - 1}{\sqrt{2} - 1}.$$

180. *To find the centre of gravity of a surface of any form.*

If, in Art. 178, we use A to denote the elementary surface PS instead of the prism Ps, we shall have

$$\text{the limit of } \frac{A}{\delta x\, \delta y} = \sqrt{1 + (d_x z)^2 + (d_y z)^2};$$

and by proceeding exactly as in that Article, we shall find

$$\bar{x} \,.\, \int_x \int_y \sqrt{1 + (d_x z)^2 + (d_y z)^2} = \int_x \int_y \{x \sqrt{1 + (d_x z)^2 + (d_y z)^2}\},$$

$$\bar{y} \,.\, \int_x \int_y \sqrt{1 + (d_x z)^2 + (d_y z)^2} = \int_x \int_y \{y \sqrt{1 + (d_x z)^2 + (d_y z)^2}\},$$

$$\bar{z} \,.\, \int_x \int_y \sqrt{1 + (d_x z)^2 + (d_y z)^2} = \int_x \int_y \{z \sqrt{1 + (d_x z)^2 + (d_y z)^2}\}.$$

181. *To find the centre of gravity of a curve of double curvature.*

If we use S for the length of the curve line, and δS for the length of a very small portion of it, we shall have the limit of $\frac{\delta S}{\delta x} = d_x S = \sqrt{1 + (d_x y)^2 + (d_x z)^2}$, and it will be found that

$$\bar{x} S = \int_x x \sqrt{1 + (d_x y)^2 + (d_x z)^2},$$

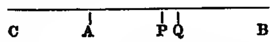
$$\bar{y} S = \int_x y \sqrt{1 + (d_x y)^2 + (d_x z)^2},$$

$$\bar{z} S = \int_x z \sqrt{1 + (d_x y)^2 + (d_x z)^2}.$$

182. We shall now add a few examples of finding the centre of gravity when the density is variable. Questions of this kind depend upon the formulæ of Art. 138, viz.—

$$\bar{x} = \frac{\Sigma\,(\rho V x)}{\Sigma(\rho V)}; \quad \bar{y} = \frac{\Sigma\,(\rho V y)}{\Sigma\,(\rho V)}; \quad \bar{z} = \frac{(\Sigma \rho V z)}{\Sigma\,(\rho V)}.$$

183. *To find the centre of gravity of a physical line, the density of which at any point varies as the* n[th] *power of its distance from a given point in the line produced.*



Let AB be the given line, and C the given point; $\mu =$ the density at a point in AB, whose distance from $C = 1$; $a = CA$, $b = CB$, $x = CP$, $\delta x = PQ$. Since a physical line is of uniform thickness throughout, we may take the length of any portion of it as the measure of the volume of that portion; hence $\delta x =$ the volume of PQ, and as the density varies as (distance from $C)^n$;

$$\therefore\ 1^n : x^n :: \mu : \mu x^n.$$

Wherefore the density at P is μx^n, and PQ is ultimately of uniform density, therefore the mass of PQ is

$$\doteq \mu x^n \delta x;$$

$$\therefore \text{ the mass of } AB = \Sigma\,(\mu x^n \delta x)$$

$$= \mu \Sigma\,(x^n \delta x)$$

$$= \mu \int x^n dx$$

$$= \mu \,.\, \frac{x^{n+1}}{n+1} + C$$

$$= \mu \,.\, \frac{b^{n+1} - a^{n+1}}{n+1},$$

between the limits $x = a$ and $x = b$.

Again, the moment of the mass of PQ about C

$$= \mu x^{n+1} \delta x;$$

$$\therefore \text{ moment of } AB \text{ about } C = \mu \int x^{n+1} dx$$

$$= \mu \,.\, \frac{x^{n+2}}{n+2} + C$$

$$= \mu \,.\, \frac{b^{n+2} - a^{n+2}}{n+2},$$

between the same limits as before.

Wherefore $\bar{x}$ being the distance of the centre of gravity of the line from C, we have

$$\bar{x} = \frac{\Sigma(\rho V x)}{\Sigma(\rho V)}$$

$$= \frac{n+1}{n+2} \cdot \frac{b^{n+2} - a^{n+2}}{b^{n+1} - a^{n+1}}.$$

REMARK. When $n = -1$,

$$\mu \int_x x^n = \mu \int_x \frac{1}{x}$$

$$= \mu \cdot \log_e x + C$$

$$= \mu \log_e \frac{b}{a}.$$

And $$\mu \int_x x^{n+1} = \mu (b-a);$$

$$\therefore \bar{x} = \frac{b-a}{\log_e\left(\frac{b}{a}\right)}.$$

Again, when $n = -2$,

$$\mu \int_x x^n = \mu\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \frac{\mu(b-a)}{ab};$$

and $$\mu \int x^{n+1} dx = \mu \int \frac{dx}{x}$$

$$= \mu \log_e \frac{b}{a};$$

$$\therefore \bar{x} = \frac{ab}{b-a} \cdot \log_e \frac{b}{a}.$$

Ex. 2. *To find the centre of gravity of a triangular plate, of uniform thickness, the density of which at any point varies as the* n[th] *power of its distance from a line through the vertex parallel to the base.*

Let ABC (fig. 37) be the triangle, CD a line through its vertex parallel to its base; μ the density at a point in the triangle at the distance 1 from CD; P, Q two points in AC very near each other, through which draw Pp, Qq parallel to the base; $b = AC$, $c = AB$, $x = CP$, $\delta x = PQ$, $\theta = \angle CAB = \angle ACD$.

Then the density at every point in the line $Pp = \mu (x \sin \theta)^n$, which may be ultimately taken as the density at every point of the element Pq. We may regard Pq as a parallelogram, whose base Pp

$$= \frac{xc}{b}, \text{ by similar triangles } ACB,\ PCp;$$

and whose altitude is $PQ \sin \theta = \delta x . \sin \theta$; its area, which we may take as the measure of its volume, is therefore

$$= \frac{xc}{b} . \delta x . \sin \theta;$$

and its mass

$$= \mu (x \sin \theta)^n . \frac{c}{b} . x\delta x . \sin \theta$$

$$= \frac{\mu c}{b} (x \sin \theta)^{n+1} \delta x;$$

$\therefore$ the mass of the triangle

$$= \Sigma \left\{ \frac{\mu c}{b} (x \sin \theta)^{n+1} \delta x \right\}$$

$$= \frac{\mu c}{b} . (\sin \theta)^{n+1} \int x^{n+1} dx$$

$$= \frac{\mu c}{b} . (\sin \theta)^{n+1} . \frac{x^{n+2}}{n+2} + C$$

$$= \frac{\mu c}{b} . (\sin \theta)^{n+1} . \frac{b^{n+2}}{n+2}$$

$$= \frac{\mu c}{n+2} . (b \sin \theta)^{n+1}.$$

And the moment of the element Pq about CD

$$=\frac{\mu c}{b}(x\sin\theta)^{n+1}\delta x\,.\,x\sin\theta$$

$$=\frac{\mu c}{b}(x\sin\theta)^{n+2}\delta x.$$

Therefore the moment of the triangle about CD

$$=\frac{\mu c}{b}\int(x\sin\theta)^{n+2}dx$$

$$=\frac{\mu c}{b}(\sin\theta)^{n+2}\,.\,\frac{x^{n+3}}{n+3}+C$$

$$=\frac{\mu c}{b}(\sin\theta)^{n+2}\,.\,\frac{b^{n+3}}{n+3}$$

$$=\frac{\mu c}{n+3}\,.\,(b\sin\theta)^{n+2}.$$

Wherefore, if a line passing through the centre of gravity of the triangle, parallel to the base, cut AC at a distance $\bar{x}$ from C, the distance of the centre of gravity from CD will be $\bar{x}\sin\theta$, and

$$\therefore\ \bar{x}\sin\theta=\frac{\dfrac{\mu c}{n+3}\,.\,(b\sin\theta)^{n+2}}{\dfrac{\mu c}{n+2}\,.\,(b\sin\theta)^{n+1}}$$

$$=\frac{n+2}{n+3}\,.\,b\sin\theta;$$

$$\therefore\ \bar{x}=\frac{n+2}{n+3}\,.\,AC.$$

And if CE be drawn bisecting the base, the centre of gravity must be in that line; hence we have two lines passing through the centre of gravity, and consequently it is the point of their intersection.

Ex. 3. *To find the centre of gravity of a quadrant of a circle, the density at any point of which varies as the* n^th^ *power of its distance from the centre.*

Let ABC (fig. 38) be the quadrant; CD, Cd two radii making angles with CA respectively equal to θ, $\theta+\delta\theta$; $AC=a$, $CP=Cp=r$, $PQ=pq=\delta r$; $\mu=$ density at the distance 1 from the centre; therefore the density at P or $p=\mu r^n$. Now we may ultimately consider Pq as a parallelogram, whose sides are PQ and Pp, or δr and $r\delta\theta$, and its area $=r\delta r\,.\,\delta\theta$, which may be taken as the measure of its volume; and its mass

$$=\mu r^n\,.\,r\delta r\,.\,\delta\theta;$$

$$\therefore \text{ mass of the quadrant} = \textstyle\int_\theta\int_r (\mu r^{n+1}).$$

Now $$\textstyle\int_r (\mu r^{n+1}) = \dfrac{\mu}{n+2}\,.\,r^{n+2} + C$$

$$= \frac{\mu}{n+2}\,.\,a^{n+2} \text{ from } r=0 \text{ to } r=a.$$

$$\therefore \text{ mass of the quadrant} = \frac{\mu}{n+2}\,.\textstyle\int_\theta a^{n+2}$$

$$= \frac{\mu}{n+2}\,.\,a^{n+2}\theta + C, \text{ from } \theta=0 \text{ to } \theta=\frac{\pi}{2}$$

$$= \frac{\mu}{n+2}\,.\,\frac{\pi}{2}\,.\,a^{n+2}.$$

Again, the moment of Pq about CB

$$=\mu r^n\,.\,r\delta r\,.\,\delta\theta\,.\,r\cos\theta;$$

$$\therefore \text{ moment of quad. about } CB = \textstyle\int_\theta\int_r (\mu r^{n+2}\cos\theta).$$

But $$\textstyle\int_r (\mu r^{n+2}\cos\theta) = \dfrac{\mu}{n+3}\,.\,r^{n+3}\cos\theta + C$$

$$= \frac{\mu}{n+3}\,.\,a^{n+3}\cos\theta;$$

$$\therefore \text{ moment of quad. about } CB = \frac{\mu}{n+3}\,.\textstyle\int_\theta a^{n+3}\cos\theta$$

$$= \frac{\mu}{n+3}\,.\,a^{n+3}\sin\theta + C$$

$$= \frac{\mu}{n+3}\,.\,a^{n+3},$$

between the same limits as before;

$$\therefore \bar{x} = \frac{\frac{\mu}{n+3} \cdot a^{n+3}}{\frac{\mu}{n+2} \cdot \frac{\pi}{2} \cdot a^{n+2}}$$

$$= \frac{n+2}{n+3} \cdot \frac{2a}{\pi}.$$

And it is manifest, from the symmetrical form of the figure, with regard to CA and CB, that $\bar{y} = \bar{x}$.

Ex. 4. *A sector of a circle* ACB (fig. 39) *revolves round one of its radii* AC *through a given angle* (β), *and generates a solid, the density at any point of which varies as the* (n)th *power of its distance from the centre* C; *to find the centre of gravity of the solid.*

Since the solid is perfectly symmetrical with regard to a plane passing through AC, and bisecting the angle β, the centre of gravity must be in that plane. Let CA be the axis of x, and a line in the plane BCA at right angles to AC, the axis of y; the axis of z being at right angles to both;

$$\therefore \bar{z} = \bar{y} \tan \frac{\beta}{2}.$$

Let $a = AC$, $\alpha = \angle BCA$, $\theta = ECA$, $\delta\theta = FCE$, $CP = Cp = r$, $PQ = pq = \delta r$, $\mu =$ the density at the distance 1 from C. Then the area of the parallelogram Qp

$$= r\delta\theta \cdot \delta r;$$

and when the sector revolves about AC, this parallelogram generates a volume

$$= r \sin \theta \cdot \beta \cdot r\delta\theta \cdot \delta r$$

$$= \beta r^2 \delta r \cdot \sin \theta \delta\theta;$$

for P's distance from AC is $r \sin \theta$, and in revolving through the angle β, the length of its path is $r \sin \theta \cdot \beta$. The density of this volume

$$= \mu r^n,$$

and therefore the mass of the element generated by Qp

$$= \mu r^n . \beta r^2 \delta r . \sin\theta . \delta\theta ;$$

$$\therefore \text{ the mass of the solid} = \mu\beta \int_\theta \int_r r^{n+2} \sin\theta.$$

But $\int_r r^{n+2} \sin\theta = \frac{r^{n+3}}{n+3} . \sin\theta + C$

$$= \frac{a^{n+3}}{n+3} . \sin\theta,$$

from $r = 0$ to $r = a$;

$$\therefore \text{ the mass of the solid} = \frac{\mu\beta}{n+3} . a^{n+3} \int_\theta \sin\theta$$

$$= -\frac{\mu\beta}{n+3} . a^{n+3} \cos\theta + C$$

$$= \frac{\mu\beta}{n+3} . a^{n+3} (1 - \cos\alpha)$$

$$= \frac{2\mu\beta}{n+3} . a^{n+3} \sin^2\frac{d}{2},$$

from $\theta = 0$ to $\theta = \alpha$.

Again, the moment of the elementary mass with respect to the plane yz

$$= \mu\beta r^{n+2} \sin\theta . \delta r . \delta\theta . r\cos\theta ;$$

$$\therefore \text{ the moment of solid} = \mu\beta \int_\theta \int_r (r^{n+3} \sin\theta \cos\theta)$$

$$= \frac{\mu\beta}{n+4} . a^{n+4} \int_\theta (\sin\theta\cos\theta)$$

$$= \tfrac{1}{2} . \frac{\mu\beta}{n+4} . a^{n+4} \sin^2\theta + C$$

$$= \tfrac{1}{2} . \frac{\mu\beta}{n+4} . a^{n+4} \sin^2\alpha ;$$

$$\therefore \bar{x} = \tfrac{1}{4} . \frac{n+3}{n+4} . a . \frac{\sin^2\alpha}{\sin^2\frac{\alpha}{2}}$$

$$= \frac{n+3}{n+4} . a\cos^2\frac{\alpha}{2}.$$

In order to find $\bar{z}$, we must divide the volume generated by the revolution of the parallelogram Pq into elements; to this end, let there be two planes passing through AC and inclined to the plane BCA, at the angles ϕ and $\phi + \delta\phi$ respectively; then the portion comprehended between them will be equal to the volume generated by Pq, in revolving through an angle $\delta\phi$, and therefore is

$$= r \sin\theta \, . \, \delta\phi \, . \, r\delta\theta \, . \, \delta r$$

$$= r^2 \delta r \, . \sin\theta\delta\theta \, . \, \delta\phi.$$

And the density of this element is μr^n, and therefore its mass is

$$\mu r^{n+2} \delta r \, . \sin\theta\delta\theta \, . \, \delta\phi,$$

and its distance from the plane ABC is $r \sin\theta \, . \sin\phi$, as is evident from the construction; and therefore its moment with respect to the plane xy

$$= \mu r^{n+3} \delta r \, . \sin^2\theta\delta\theta \, . \sin\phi\delta\phi;$$

therefore the moment of the solid with respect to the plane xy

$$= \mu \int_r \int_\theta \int_\phi (r^{n+3} \sin^2\theta \sin\phi)$$

$$= \frac{\mu a^{n+4}}{n+4} \, . \int_\theta \int_\phi (\sin^2\theta \sin\phi)$$

$$= \frac{\mu a^{n+4}}{n+4} \int_\theta (-\sin^2\theta \cos\phi + C)$$

$$= \frac{\mu a^{n+4}}{n+4} \int_\theta (1 - \cos\beta) \sin^2\theta,$$

taken from $\phi = 0$ to $\phi = \beta$.

Now $\int_\theta \sin^2\theta = \frac{1}{2} \int_\theta (1 - \cos 2\theta)$

$$= \tfrac{1}{2} (\theta - \tfrac{1}{2} \sin 2\theta) + C$$

$$= \frac{\alpha}{2} - \frac{\sin 2\alpha}{4},$$

taken from $\theta = 0$ to $\theta = \alpha$;

therefore the moment of the solid with respect to the plane xy

$$= \frac{\mu a^{n+4}}{n+4} (1 - \cos\beta) \tfrac{1}{2}(\alpha - \tfrac{1}{2}\sin 2\alpha)$$

$$= \frac{\mu a^{n+4}}{n+4} \sin^2\frac{\beta}{2} (\alpha - \sin\alpha\cos\alpha);$$

$$\therefore \bar{z} = \frac{n+3}{n+4} \cdot \frac{a}{2} \cdot \frac{\sin^2\frac{\beta}{2}}{\sin^2\frac{\alpha}{2}} \frac{\alpha - \sin\alpha\cos\alpha}{\beta};$$

and therefore $\bar{y} = \bar{z}\cot\dfrac{\beta}{2}$

$$= \frac{n+3}{n+4} \cdot \frac{a}{2} \cdot \frac{\cos\frac{\beta}{2}\sin\frac{\beta}{2}}{\sin^2\frac{\alpha}{2}} \cdot \frac{\alpha - \sin\alpha\cos\alpha}{\beta}.$$

Ex. 5. Find the centre of gravity of a cone, the density at every point of which varies as the square of its distance from a plane through the vertex parallel to the base.

$$\bar{x} = \frac{5}{6} \text{ of the cone's axis.}$$

Ex. 6. Find the centre of gravity of the eighth part of a sphere, the density at any point, whose distance from the centre is r, being proportional to

$$\frac{a}{r}\sin\frac{\pi r}{2a},$$

where a denotes the radius of the sphere.

$$\bar{x} = \bar{y} = \bar{z} = a\left(\cdot 1 - \frac{2}{\pi}\right).$$

GULDIN'S PROPERTIES.

184. *The surface generated by a plane curve line, which revolves about a fixed axis, is equal to the product of the length of the curve line by the length of the path described by its centre of gravity.*

For let AB (fig. 33) be the curve line, and Ox the line about which it revolves through an angle θ; then using the same notation as in Art. 174, the point P describes an arc $= y\theta$, consequently the arc PQ describes a zone, of which the length is $y\theta$ ultimately, and the breadth $= \delta s$; hence the area of the zone is ultimately $= \theta y \delta s$; and therefore the area of the whole surface generated is

$$\doteq \Sigma\, (\theta y \delta s) = \theta \int y ds;$$

the integral being taken between the limits corresponding to

$$x = OC, \quad x = OD.$$

But if $\bar{y}$ be the distance of the centre of gravity of the arc AB from the axis Ox, we have shewn in Art. 174, that

$$\bar{y}\,.\,(\text{arc } AB) = \int y ds, \text{ between the same limits;}$$

hence the surface generated

$$= \theta \bar{y}\,.\,(\text{arc } AB).$$

Now $\theta \bar{y}$ is the length of the path described by the centre of gravity, consequently the last equation expresses the property to be proved.

185. *The volume generated by a plane area, revolving about a fixed axis in its own plane, is equal to the product of the area into the length of the path described by its centre of gravity.*

Let A be the revolving area; δA a portion of it so small that it may be all considered to be at the same distane y from the axis. Then if θ be the angle through which the area revolves, δA will describe a volume which may be considered to be a thin cylinder bent into the form of a portion of a ring. The area of the base of this cylinder is δA, and its length is $y\theta$, consequently the volume generated by δA

$$= \theta y \delta A;$$

and therefore the whole volume generated

$$= \theta \Sigma (y \delta A).$$

But if $\bar{y}$ be the distance of the centre of gravity of the area A from the fixed axis, we have from the nature of the centre of gravity

$$\Sigma (\delta A) \,.\, \bar{y} = \Sigma (y \delta A),$$

$$\text{or } A\bar{y} = \Sigma (y \delta A);$$

hence the whole volume generated

$$= \theta \bar{y} A :$$

an equation which expresses the property which was to be proved.

REMARK. If the curve line in Art. 184, or the plane area in Art. 185, does not revolve about a *fixed* axis during its whole motion but moves in any such manner that it may at any moment be assumed to be revolving for an instant about a fixed axis in its plane; then the propositions in those articles will be true for *each instant;* and consequently, by adding these results together, *those articles will be true* for the *whole* motion whatever be the nature of the path of the centre of gravity. But it is necessary to notice that when the instantaneous axis, about which the generating curve or area is supposed to be revolving, is in such a position that the instantaneous axis divides the curve or area into two portions, the part generated by one of those portions during that instant is to be considered positive, and that generated by the other negative, and the propositions fail in this case. *As long therefore as the line of instantaneous revolution lies entirely out of the limits of the generating curve or area,* the propositions in Arts. 184, 185 hold true, viz.:

The surface generated by a plane curve line which moves in any manner (subject to the limitations just named), *is equal to the product of the length of the curve line by the length of the path described by its centre of gravity.* And

The volume generated by a plane area, which moves in any manner (subject to the same limitations), *is equal to the product of the area into the length of the path described by its centre of gravity.*

CHAPTER VII.

ON MECHANICAL INSTRUMENTS.

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186. EVERY machine, how complicated soever its construction, is found to be reducible to a set of simple ones, called the *Mechanical Powers.* These, though authors differ considerably on the subject, are generally said to be six in number, viz.:

1. The Lever;
2. The Pulley;
3. The Wheel and Axle;
4. The Inclined Plane;
5. The Screw;
6. The Wedge.

These are not the most simple machines; for, *rods* used in pushing, and *cords* used in pulling, are much more simple; in fact, every machine will be found to be a combination of levers, cords, and inclined planes, and these might consequently be called the simple Mechanical Powers, with much greater propriety than the six before mentioned. As, however, these are not very complicated in construction and application, and as levers, cords, and inclined planes do always, in actual practice, present themselves in machinery, in one or more of these six combinations, it will very much facilitate our enquiries into the properties of any proposed machine, to be acquainted with their forms and the advantages to be expected from their use.

In speaking of any machine, the force which is applied to work it is called the *Working Power*, or simply, the *Power;* the weight to be raised, or resistance to be overcome, is called the *Weight;* the point where the machine is applied to produce its effect is called the *Working Point;* and the fraction

$$\frac{\text{Weight}}{\text{Power}}$$

is called the *Mechanical Advantage* (by some authors the *Power*, but this creates confusion by confounding it with the former definition of power) of the machine.

187. Every machine is useless until put in motion, and therefore its parts ought to be so arranged and adapted that the given power may be able to overcome the proposed weight, and move it with the requisite degree of celerity; but, in discussing the theory of the Mechanical Powers, it will be sufficient to determine the ratio of the weight to the power when they balance each other, for then the slightest addition made to the power will cause it to preponderate and put the machine in motion.

188. It is very important to remark, that when a power is employed in working a machine, a very considerable portion of it is found not to reach the working point, being spent in overcoming the stiffness of the cords and the roughness of surfaces which rub against each other. Much power is also lost through the imperfection of workmanship, the bending of rods, beams and other materials, which are intended to be rigid, the resistance of the air, &c.; but the introduction of the consideration of these things, though very important in a practical point of view, would only tend to embarrass the student by rendering our investigations tedious and perplexing. We shall therefore at first suppose cords to be perfectly flexible, surfaces quite smooth, workmanship geometrically exact, rods and beams perfectly rigid, the air to offer no resistance; &c.

"It is scarcely necessary to state, that, all these suppositions being false, none of the consequences deduced from them can be true. Nevertheless, as it is the business of Art to bring machines as near to this state of ideal perfection as possible, the conclusions which are thus obtained, though false in a strict sense, yet deviate from the truth in but a small degree. Like the first outline of a picture, they resemble in their general features that truth, to which, after many subsequent corrections, they must finally approximate.

"After a first approximation has been made on the several suppositions which have been mentioned, various effects, which have been previously neglected, are successively taken into account. Roughness, rigidity, imperfect flexibility, the resistance of air and other fluids, the effects of the weight and inertia of the machine, are severally examined, and their laws and properties detected. The modifications and corrections thus suggested, as necessary to be introduced into our former conclusions, are applied, and a second approximation, but still *only* an approximation to truth is made. For, in investigating the laws which regulate the several effects just mentioned, we are compelled to proceed upon a new group of false suppositions. To determine the laws which regulate the friction of surfaces, it is necessary to assume that every part of the surfaces of contact is uniformly rough; that the solid parts which are imperfectly rigid, and the cords which are imperfectly flexible, are constituted throughout their entire dimensions of a uniform material; so that the imperfection does not prevail more in one part than another. Thus all irregularity is left out of account, and a general average of the effects taken. It is obvious therefore, that by these means we have still failed in obtaining a result exactly conformable to the real state of things; but it is equally obvious, that we have obtained one much more conformable to that state than had been previously accomplished, and sufficiently near it for most practical purposes.

"This apparent imperfection in our instruments and powers of investigation, is not peculiar to Mechanics; it pervades all departments of natural science. In Astronomy, the motions of the celestial bodies, and their various changes and appearances, as developed by theory, assisted by observation and experience, are only approximations to the real motions and appearances which take place in nature. It is true that these approximations are susceptible of almost unlimited accuracy; but still they are, and ever will continue to be, only approximations. Optics, and all other branches of natural science, are liable to the same observations*."

* Captain Kater's Treatise on Machines.

I. *On the Lever.*

189. Def. A *Lever* is a rigid rod straight or bent, moveable in a certain plane about one of its points, which is fixed and called its fulcrum.

190. *In a lever when there is equilibrium the power and weight are to each other inversely as the perpendiculars from the fulcrum upon the directions in which they act.*

(Both the power and weight are supposed to act in the plane in which the lever is moveable, which is technically called the plane of the lever).

Let AB (figs. 40, 42) or AC (fig. 41) or BC (fig. 43), be a lever whose fulcrum is C; A, B the points at which the power P and weight W act; CY, CZ perpendiculars from C upon their directions. Then the equilibrium will not be disturbed by applying at C two forces P', P'' parallel and equal to P, and two others W', W'' parallel and equal to W. We have thus, six forces acting on the lever, of which (P, P'') and (W, W'') form two couples, and the two remaining forces P', W' being counterbalanced by the reaction of the fulcrum, may be removed. Hence the couple (P, P'') whose arm is CY, balances the couple (W, W'') whose arm is CZ, consequently their moments must be equal;

$$\therefore P \cdot CY = W \cdot CZ.$$

191. *To find the pressure on the fulcrum* C.

We have shewn that P and W are equivalent to two forces P', W' acting at C, and two equal couples (P, P''), (W, W''); these couples may be removed because they are equal and opposite and therefore balance each other. It appears, then, that P and W are equivalent to P' and W' acting at C. Consequently the pressure on the fulcrum is the same as if the power and weight were both transposed to it parallel to themselves.

192. We have considered the weight of the lever inconsiderable when compared with P and W, but if this should not

be the case, let w be its weight, G its centre of gravity. Then we may suppose the whole force w, which gravity exerts upon the lever, to be applied at G; this force may be converted into a couple whose moment is $w \cdot CG$, and as there is equilibrium between the three couples, the sum of the moments of the two which act in one direction (*i. e.* positive or negative) must be equal that of the third;

$$\therefore P \cdot CY + w \cdot CG = W \cdot CZ,$$

is the equation of equilibrium in this case.

193. Examples of levers of the same kind as the one in fig. 40, are the common balance, steelyards, pokers, &c.; and scissors, pincers, &c. are instances of two such levers having a common fulcrum.

Examples of levers of the same kind as those in figs. 41, 43, are the oars and rudders of boats, cutting-knives moveable about one end, &c.; and tongs, sheep-shears, &c. are instances of the combination of two such levers with a common fulcrum.

Examples of the bent lever, in fig. 42, are gavelocks, jemmies, bones of all animals, &c.

194. We have defined a lever to be a rigid rod, but we may consider any rigid body having a fixed axis as a compound lever, whose fulcrum is the axis; and if powers $P_1, P_2, P_3 \ldots P_n$, act upon this lever, and balance the weights $W_1, W_2, W_3 \ldots\ldots W_m$, then

$$P_1p_1 + P_2p_2 + \ldots\ldots + P_np_n = W_1w_1 + W_2w_2 + \ldots\ldots + W_mw_m,$$

$$\text{or } \Sigma(Pp) = \Sigma(Ww);$$

the powers and weights being supposed to act in planes at right angles to the axis, and $p_1, p_2 \ldots p_n$; $w_1, w_2 \ldots w_m$ being the respective perpendiculars from the axis upon the directions in which the powers and weights act.

This may be proved as before, by converting the powers and weights into couples, and then transposing them into one plane; and it will also appear, that the pressure on the axis or fulcrum is the same as it would be if all the forces were transported in their own planes parallel to themselves to the axis.

II. *On the Pulley.*

195. DEF. *A Pulley* is a wheel of wood or metal, turning on an axis through its centre at right angles to its plane, and usually enclosed in a frame or case, called its *block*, which admits a rope to pass freely over the circumference of the pulley, in which there is usually a groove to receive it and prevent its slipping out. The pulley is said to be *fixed* or *moveable*, according as its axis is stationary or not. An assemblage of several pulleys is called a *system of pulleys*.

196. It will be necessary before investigating the properties of the pulley to premise, that if a cord be stretched by two equal forces applied at its extremities in contrary directions, there will be a tendency to break; the force which the rope, in consequence of the cohesion of its particles, exerts to resist this tendency, must be equal and opposite to that which causes the tendency; it is called the *tension* of the rope. Hence tension is a force which is exerted equally in every part, tends from the extremities of a cord towards the middle, and is always equal to either of the equal forces, by which the cord is stretched. If one end of the cord, instead of being acted on by a force, be fastened to a fixed point, the tension will not be altered; for the fixed point will, by its reaction, exactly supply the place of the force.

197. *In the single fixed pulley when there is equilibrium the power and weight are equal.*

Let ABK (fig. 44) be the pulley, C its centre, CN its block; P and W the power and weight acting at the extremities of the cord passing over the pulley, and having the part AB in contact with it. Then we may consider the pulley ABK as a lever whose fulcrum is C; and therefore drawing the radii CA, CB to the points A and B, we have

$$P \cdot CA = W \cdot CB;$$

$$\therefore P = W.$$

Hence it appears that no mechanical *advantage* is gained by the use of this pulley; the only purpose for which it is used is to change the direction in which a force is transmitted.

198. *To determine the pressure on the fulcrum* C.

Transpose the forces P and W to that point, and put θ for the angle at which AP and BW are inclined to each other, and let R be the pressure, which is, of course, the resultant of these transposed forces, and bisects the angle between them; hence resolving these forces in the direction of R, we find

$$R = P\cos\frac{\theta}{2} + P\cos\frac{\theta}{2}$$

$$= 2P\cos\frac{\theta}{2}.$$

This pressure is transmitted to N, the fixed point to which the block is attached.

199. *In the single moveable pulley when there is equilibrium the power is to the weight* :: 1 : 2 × *cosine of half the angle between the strings.*

Let the power P act at the extremity P of the cord $PABD$ (fig. 45), which passing under the pulley has the part AB in contact with it; and its other extremity fastened at D. The weight W hangs from the block at N.

Exactly as in the last case, we find the pressure on the centre C to be

$$2P\cos\frac{\theta}{2},$$

θ being the angle between the strings AP, BD; this force is transmitted through the block in the direction CN, bisecting the angle θ; wherefore the action of W must be equal to it and in the opposite direction, otherwise there cannot be an equilibrium;

$$\therefore\ W = 2P\cos\frac{\theta}{2},$$

$$\text{and consequently } P : W :: 1 : 2\cos\frac{\theta}{2}.$$

200. No mechanical *advantage* can be gained by the use of this pulley, unless

$$2\cos\frac{\theta}{2} > 1,$$

$$\text{and } \therefore\ \cos\frac{\theta}{2} > \tfrac{1}{2} > \cos 60^\circ;$$

$$\therefore\ \theta < 120^\circ;$$

that is, unless the strings are inclined to each other at a less angle than 120°.

The greatest possible *advantage* will be gained when the strings are parallel, for then $\theta = 0$, and $\cos\frac{\theta}{2} = 1$,

$$\text{and therefore } W = 2P.$$

201. If the weight of the pulley and its block be considerable, it must be considered as an additional weight, and added to W in the above expressions.

202. *To find the conditions of equilibrium in a system of pulleys, where each pulley hangs by a separate string, the strings being all parallel.*

Let A_1, A_2, A_3, ... (fig. 46) be the pulleys; M_1, M_2, M_3... the points where the strings are fastened to an immoveable block. Then P is the tension of the string passing under A_1. The two strings A_1P, A_1M_1 have to support the tension of N_1A_2; so N_1A_2 and M_2A_2 support that of N_2A_3, and so on; therefore,

$$(P=)\text{ tension of } A_1P : \text{tension of } N_1A_2 :: 1 : 2,$$
$$\text{tension of } N_1A_2 : \text{tension of } N_2A_3 :: 1 : 2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\text{tension of } N_2A_3 : \text{tension of } N_3W (= W) :: 1 : 2;$$

$$\therefore\ P : W :: 1 \times 1 \times 1 \times \dots\dots : 2 \times 2 \times 2 \dots\dots$$

If n be the number of moveable pulleys, then

$$P : W :: 1^n : 2^n;$$

$$\therefore\ W = 2^n P.$$

203. If the weights of the pulleys and blocks are considerable, let A_1, A_2, A_3 ... represent the weights of the pulleys and blocks denoted by those letters in the figure; and let T_1, T_2 ... be the tensions of the strings N_1A_2, N_2A_3 .... Then, as before, the weights of the pulleys must be added to the tensions of the respective cords which they support;

$$\therefore\ P : T_1 + A_1 :: 1 : 2;$$

$$\therefore\ T_1 = 2P - A_1.$$

$$\text{Similarly,}\quad T_2 = 2T_1 - A_2$$
$$= 2^2P - 2A_1 - A_2,$$
$$T_3 = 2T_2 - A_3$$
$$= 2^3P - 2^2A_1 - 2A_2 - A_3,$$

and so on, the law being manifest; then, since the tension of the last string $= W$, we have

$$W = 2^nP - 2^{n-1}A_1 - 2^{n-2}A_2 - 2^{n-3}A_3 - \ldots\ldots - A_n.$$

It appears from this expression, that the weights of the pulleys diminish the advantage of this system.

204. If all the pulleys are equal, then

$$W = 2^nP - A_1(2^{n-1} + 2^{n-2} + \ldots\ldots + 1)$$
$$= 2^nP - A_1\frac{2^n - 1}{2 - 1}$$
$$= 2^nP - (2^n - 1)A_1$$
$$= 2^n(P - A_1) + A_1;$$
$$\therefore\ W - A_1 = 2^n(P - A_1).$$

Hence, if we suppose both the power and weight diminished by the weight of a pulley, we may then neglect the consideration of the heaviness of the pulleys.

205. In the system (fig. 47) where each string is attached to the weight, let T_1, T_2... be the tensions of the first, second ...strings; then if the weights of the pulleys are inconsiderable, we have

$$T_1 = P,$$
$$T_2 = 2T_1 = 2P,$$
$$T_3 = 2T_2 = 2^2P,$$
$$T_4 = 2T_3 = 2^3P;$$

and if there be n separate strings,

$$T_n = 2^{n-1}P.$$

Now W is supported by the tensions of the n strings fastened to the block B, and

$$\therefore\ W = T_1 + T_2 + \ldots\ldots + T_n$$
$$= P(1 + 2 + 2^2 + \ldots 2^{n-1})$$
$$= P.\frac{2^n - 1}{2 - 1}$$
$$= P(2^n - 1).$$

206. In the system (fig. 48), let T_1, $T_2\ldots$ be the tensions of the first, second... strings; then $T_1 = P$; and T_2 has to support three tensions equal to P; therefore

$$T_1 = P,$$
$$T_2 = 3T_1 = 3P,$$
$$T_3 = 3T_2 = 3^2P,$$
$$T_4 = 3T_3 = 3^3P;$$

and if there be (n) different strings, the tension of the last is

$$T_n = 3^{n-1}P.$$

Now the weight W is supported by two strings whose tensions are T_1, two of which the tensions are T_2, &c.;

$$\therefore\ W = 2T_1 + 2T_2 + \ldots\ldots + 2T_n$$
$$= 2P.(1 + 3 + 3^2 + \ldots\ldots 3^{n-1})$$
$$= 2P.\frac{3^n - 1}{3 - 1}$$
$$= P.(3^n - 1).$$

REMARK. If the weights of the pulleys and blocks are not inconsiderable, they may be taken into account, in this and every other system, by adding each to the tension of that string which supports it, as in Art. 203.

207. In the system, fig. 49, the weight W is supported by the tensions of all the strings at the lower block, and as it is the same string which passes round all the pulleys, the tension of every part $=P$; wherefore, if there be n pulleys in the lower block, there are $2n$ strings supporting the weight, and therefore

$$W = 2nP.$$

III. *On the Wheel and Axle.*

208. The wheel and axle consists of a cylinder and a wheel firmly attached to each other, and being moveable about a fixed axis coinciding with the axis of the cylinder, and passing through the centre of the wheel at right angles to its plane, as in fig. 50.

The power P acts by means of a cord wrapped round the circumference of the wheel C, and the weight W is fastened to a cord which is wound upon the cylinder AB as P turns the machine round its axis; and thus W is raised.

209. *To find the condition of equilibrium on the wheel and axle.*

We may consider P and W as forces acting upon a rigid body with a fixed axis, and therefore their moments about that axis must be equal;

$$\therefore\ P \times \text{(perpendicular upon its direction from the axis)},$$
$$= W.\text{(perpendicular upon its direction from the axis)}.$$

Now these perpendiculars are respectively the radii of the wheel and of the cylinder;

$$\therefore\ P.\text{(radius of the wheel)} = W.\text{(radius of the axle)}.$$

210. If the thickness of the rope be considerable, it must be taken into account.

We may suppose the actions of P and W to be transmitted along the middle or axis of the rope, and then the perpendiculars upon the directions of P and W will be respectively equal to

radius of wheel + radius of rope,

and radius of axle + radius of rope,

and the condition of equilibrium is

$$P\,.\,(\text{rad. wheel} + \text{rad. of rope}) = W\,(\text{rad. axle} + \text{rad. of rope}).$$

This diminishes the *advantage* of the machine.

211. The pressure on the axis of this machine may be found by transposing P and W in their own planes, parallel to themselves, to the axis.

IV. *On the Inclined Plane.*

212. This machine is nothing more than a plane inclined to the horizon. The condition of equilibrium may be thus found.

Let AB (fig. 51) be the plane; AC parallel and BC perpendicular to the horizon; W the weight, P the power. Draw WR perpendicular to the plane, WG perpendicular to the horizon. P is supposed to act in the plane RWB. The weight W is kept at rest by three forces, viz. P in the direction WP: gravity $(= W)$ in the direction WG, and reaction R of the plane in the direction WR.

Denote the angle PWB by θ, and the inclination BAC of the plane to the horizon by i; and resolve the three forces, acting on the point W, in a direction parallel to the planes; the sum will be

$$P\cos PWB - W\cos AWG + R\cos RWB$$
$$= P\cos\theta - W\,.\,\sin i.$$

But since there is an equilibrium, this sum must be equal to zero,

$$\therefore\; P\cos\theta = W\sin i,$$

which is the condition of equilibrium.

213. If P's direction should happen to be parallel to the plane, $\theta = 0$ and $\cos\theta = 1$;

$$\therefore\ P = W\sin i.$$

But if P's direction should happen to be parallel to the horizon, $\theta = -i$ and $\cos(-i) = \cos i$;

$$\therefore\ P\cos i = W\sin i;$$

$$\therefore\ P = W\tan i.$$

214. *To find the reaction of the plane.*

Resolve the forces in a direction at right angles to that in which P acts:

$$\therefore\ R\sin RWP - W\sin GWP = 0,$$

$$\text{or}\quad R\cos\theta - W\sin(90 + i + \theta) = 0;$$

$$\therefore\ R = W.\frac{\cos(i+\theta)}{\cos\theta}.$$

V. *On the Screw.*

215. This mechanical power is a combination of the lever and inclined plane; it may be conceived to be thus generated.

Let $ABCD$ (fig. 52) be a cylinder; $BEFC$ a rectangle whose base BE is equal to the circumference of the cylinder. Divide this rectangle into any convenient number of equal rectangles GE, IH, CK; and draw their diagonals BH, GK, IF. Then, if this rectangle CE be wrapped upon the cylinder, so that BE coincides with the circumference of the base, E, H, K, F will respectively fall upon the points B, G, I, C of the cylinder, and the lines BH, GK, IF will trace out upon its surface a continuous spiral thread $BLGMINC$ winding uniformly up the cylinder. The cylinder is usually made protuberant where the spiral line $BLGMINC$ falls upon it so that the thread becomes a winding inclined plane, projecting from the cylinder as in fig. 53, and differing from the inclined plane

*BH** in nothing but its winding course. This is the external screw. The internal screw is formed by applying the parallelogram *BEFC* to a hollow cylinder, equal to the former, and making a groove where the thread falls to fit the protuberant thread of the external screw. This internal screw is often called a *nut*, and the other the *screw*. When the two screws are thus adapted to each other, the external or the internal screw, as the case requires, may be moved by means of a lever about their common axis, as in figs. 54, 55. The force being applied to the lever at right angles to it, in a plane parallel to the base of the cylinder.

The screw and nut thus applied to each other, resemble two inclined planes, such as *BHG* and *HBE*, one of which is laid upon, and slides down the other; and as the planes wind round the cylinder a rotatory motion ensues. When the machine is worked, the weight is laid upon the nut, and thus causes its inclined plane to press upon that of the screw in the direction of gravity. The consequence would be, that the nut and weight with it would begin to slide down the thread of the screw and descend, but this is prevented by confining the nut so that it cannot have a rotatory motion, but only one of ascent or descent. The screw is then turned round by means of a lever passing through its head, and thus its inclined thread sliding under that of the nut, forces the nut and the weight upon it to ascend, just as by pushing the inclined plane *EBH* in the direction *EB*, the

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* The following illustration renders this very clear:—

"When a road directly ascends the side of a hill, it is to be considered as an inclined plane; but it will not lose this mechanical character, if, instead of directly ascending towards the top of the hill, it winds successively round it, and gradually ascends so as after revolutions to reach the top. In the same manner a path may be conceived to surround a pillar by which the ascent may be facilitated upon the principle of the inclined plane. Winding stairs constructed in the interior of great columns partake of this character; for although the ascent be produced by successive steps, yet if a floor could be made sufficiently rough to prevent the feet from slipping, the ascent would be accomplished with equal facility. In such a case the winding path would be equivalent to an inclined plane, bent into such a form as to accommodate it to the peculiar circumstances in which it would be required to be used. It will not be difficult to trace the resemblance between such an adaptation of the inclined plane and the appearances presented by the thread of the *screw;* and it may hence be easily understood that a screw is nothing more than an inclined plane, constructed upon the surface of a cylinder."—Captain Kater's *Machines*.

plane GBH would be made to ascend. One turn of the screw raises the weight through an altitude equal to the distance between two threads. Sometimes, however, the nut is firmly fixed so as to admit of no motion whatever (as in fig. 54); and then the thread of the screw, in sliding under that of the nut, forces the screw to descend and press violently against any obstacle which may be opposed to it. In some cases the weight is not applied to the nut, but to the screw; but as the two inclined planes are perfectly equal and similar, it will require the same force to support a weight on one as on the other, and for this reason one investigation will serve for both.

As before observed, the screw is worked by applying a power P at the end of a lever; and the moment of P to turn the screw round

$$= P \times \text{length of the lever},$$

and therefore P is equivalent to a force

$$\frac{P \times \text{length of the lever}}{\text{rad. cylinder}}$$

acting immediately at the thread of the screw in a horizontal direction parallel to that in which P acts. Now the inclined plane on which W rests, by means of the nut, is only BH wrapped round the cylinder; its inclination to the horizon or base of the cylinder is therefore HBE.

Hence we have

$$P \times \frac{\text{length of lever}}{\text{rad. of cylinder}} = W . \tan HBE, \quad \text{(Art. 213)}$$

$$= W . \frac{HE}{BE}$$

$$= W . \frac{\text{distance between two threads}}{\text{circumf. of cylinder}}.$$

But the radii of circles are proportional to their circumferences;

$$\therefore \frac{\text{length of lever}}{\text{rad. of cylinder}} = \frac{\text{circumf. described by power}}{\text{circumf. of cylinder}};$$

$$\therefore P \cdot \frac{\text{circumf. by power}}{\text{circumf. of cylinder}} = W \cdot \frac{\text{dist. between two threads}}{\text{circumf. of cylinder}};$$

$$\therefore P = W \cdot \frac{\text{dist. between two threads}}{\text{circumf. described by power}}.$$

As the distance between two successive threads can be made very small, and the circumference described by the power as large as we please, the advantage of this machine is very great; and it is remarkable, that it does not depend upon the thickness of the screw.

VI. *On the Wedge.*

216. A wedge is the solid figure defined by Euclid (Book XII. Def. 4) as a triangular prism. Its two ends are equal and similar triangles, and its three sides rectangular parallelograms (see fig. 56). It is principally used in splitting timber, and separating bodies which are very strongly united, and in raising very heavy weights through a small altitude, for the purpose of introducing a lever, or some other more convenient machine. *AB* is called its *edge*, *CDEF* its *head*, *CABD* and *FABE* its *faces*.

When used, its edge is introduced into a small cleft prepared to receive it, and then by violent blows with a hammer on its head its body is driven between the substances, which are thus separated by an interval equal to the breadth of the head. After this, a larger wedge may be introduced, if necessary, and treated as before, until the requisite degree of separation is effected.

As the wedge is driven in by violent blows, if its sides were perfectly smooth it would start back by the pressure of the obstacles upon them in the interval between the strokes; and thus we should fail in effecting and maintaining the requisite degree of separation, and the machine would be rendered useless. In practice, however, the friction in this machine is always so great as to prevent any recoil, and forms, in fact, the principal resistance to be overcome in driving the wedge. The mode of working this machine will at once present itself to the reader as being *totally different in principle* from that of all the other machines we have described. These are made to work by the *constant* and *steady* exertion of a power, uniformly pressing upon that point of the machine at which it is applied, and *gradually* producing motion in the weight; but in this machine motion is accumulated in a hammer, by suffering it to descend from an altitude, and is *suddenly* by an *impulse* transferred to the wedge. In this case it must evidently be a useless labour to attempt to calculate the ratio of P to W, when they act by pressures, as in the other mechanical powers, and are in equilibrium. It is true, when we know this ratio, a slight increase* of P will gradually produce a motion in W, and thus separate the obstacles; but this mode of working the machine is so widely different from that actually practised, that it would be a waste of time and labour to attempt an explication on Statical principles. A slight stroke with a hammer is found to be far more effective than several tons of pressure. The only theoretical property of the wedge which agrees with practice is that its *advantage* is increased by diminishing its angle DBE.

All cutting instruments, such as knives, swords, hatchets, chisels, planes used by carpenters, nails, pins, needles, &c. are modifications of the wedge. Of these, knives, planes, pins and needles, are usually worked by pressure, but swords, hatchets, chisels, nails, &c. are worked by percussion.

GENERAL PROPERTY OF MACHINES.

217. If the nature of a machine be such, that when the power and weight balance each other in one position of the machine they will balance in every position of it, a very remarkable property appertains to it, deducible from the principle of virtual velocities, which we may state as follows:

The power is to the weight as the space moved through by the weight when the machine is put in motion is to the space moved

* This, however, supposes the sides to be perfectly smooth, for otherwise the friction itself, without the assistance of any power at all, would preserve the equilibrium.

through by the power in the same time; the spaces being measured respectively in the directions in which the power and weight act.

Let the whole space (measured thus) through which the power P moves be divided into a very large number of spaces s_1, s_2..., and let s'_1, s'_2... be the corresponding spaces described by the weight W; then

$$s = s_1 + s_2 + \ldots\ldots$$

$$s' = s'_1 + s'_2 + \ldots\ldots$$

But because P and W are always in a position of equilibrium; s_1, s'_1, are their virtual velocities for the first position;

$$\therefore\ Ps_1 + Ws'_1 = 0,$$

$$\text{Similarly}\ Ps_2 + Ws'_2 = 0, \text{ for the 2nd position}$$

$$Ps_3 + Ws'_3 = 0, \ \ldots\ldots \text{ 3rd } \ldots\ldots$$

$$\ldots\ldots\ldots\ldots = \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore\ Ps + Ws' = 0;$$

$$\therefore\ \frac{P}{W} = -\frac{s'}{s}.$$

This equation expresses the property enunciated. The negative sign points to the fact, that the direction of the action of one of the two forces P, W is opposed to the direction in which the point moves on which it acts.

Mechanical powers possessing this property are;—

(1) The straight lever supporting weights.

(2) All the pulleys in which the strings are parallel.

(3) The Wheel and Axle.

(4) The Screw.

(5) The Inclined Plane, only when the Power hangs by a string passing over the top of the plane.

WHITE'S PULLEY.

218. In the common systems of pulleys each pulley has its own independent centre of motion; and consequently as they

all move with different velocities and with different degrees of pressure, some of them will be liable to greater wear than others, which will very much tend to increase the friction and other inequalities and resistances; and will greatly diminish the efficiency of the machine. To obviate these difficulties, Mr James White invented a system of pulleys (fig. 57), consisting of two blocks A, B, into which grooves were cut, the radii of those in the upper block being as the numbers 1, 3, 5... and the radii of those in the lower block being as the numbers 2, 4, 6... Now, suppose the lower block to be raised through one inch, then each of its strings will be shortened one inch, and therefore the circumference of the pulley BB_1 describes one inch; that of AA_1, two inches; that of BB_2, three inches, and so on; which numbers being proportional to the radii of the respective pulleys, they will all move with the same angular velocity; and, consequently, each block instead of being composed of separate pulleys may consist of one solid piece of wood or metal, containing the grooves before mentioned. The disadvantage of this system is, that if the cord be at all elastic it cannot be kept stretched in every part on account of the tension not being the same throughout, so that the smaller grooves are rather a hindrance to the motion than a help.

HUNTER'S SCREW.

219. We have seen (Art. 215) that the advantage of a screw increases in proportion as the distance between the threads diminishes, and as the length of the lever at which the power acts increases; therefore, by making the threads of the screw sufficiently fine, we may increase the advantage as much as we please; but there is a limit to the fineness of the threads; for as all the weight is borne upon them, if they are too fine they will not be sufficiently strong to bear the load. If we, on the other hand, increase the length of the arm of the lever, with the view of increasing the advantage of the screw, the power will have to describe an inconveniently large circle. To obviate these natural defects, and yet increase the advantage to any degree, Mr Hunter invented the screw in fig. 58; A and B are two common screws, of which A is also a hollow screw to admit B, which is fastened

to the moveable plate D of wood or metal. If D, d be the distances between two threads of the screws A, B respectively; then, while the power describes one circumference, A descends through D, and B ascends in A through d, and the space descended by the plane D is $D-d$; for when A descends it carries B along with it, though B is at the same time ascending in A. Wherefore, by Art. 217,

$$P.(\text{circumf. described by } P) = W.(D-d);$$

$$\therefore \frac{W}{P} = \frac{\text{circumf. described by } P}{D-d}.$$

Now we can make D and d as nearly equal as we please without diminishing the strength of the machine, and therefore the advantage of this screw admits of indefinite increase.

220. It appears from Art. 209, that the advantage of a wheel and axle is

$$\frac{\text{rad. of wheel}}{\text{rad. of axle}},$$

which might theoretically be augmented *ad libitum*, either by increasing the radius of the wheel, or by diminishing that of the axle. But by the former means, the power would practically have to describe an inconveniently large space, and the machine would become cumbrous; and, in the latter case, it would be too weak to bear the pressure of the weight upon its axle. To remedy these inconveniences, and at the same time to increase the advantage in any requisite degree, the form of fig. 59 has been given to it; where A is the wheel, B and C two axles of unequal radii, firmly fixed to each other, and having the same axis. The cord BDC as P descends is wound upon the axle B with the larger radius, and is at the same time unwound from the axle C with the smaller radius; it passes under a pulley D, to which the weight W is attached. Let R be the radius of the wheel, rr' those of the axles B, C. Then when the machine turns once round, P descends through $2\pi R$, and the length of the cord wound upon B is $2\pi r$, and the length unwound at the same time from C is $2\pi r'$; wherefore, upon the whole, the length of cord hanging down from the axles is diminished by

$$2\pi r - 2\pi r';$$

and, therefore, W has ascended through

$$\pi r - \pi r'.$$

Wherefore, by Art. 217,

$$P : W :: \pi r - \pi r' : 2\pi R,$$
$$:: r - r' : 2R;$$

$$\therefore \text{ the advantage} = \frac{W}{P} = \frac{2R}{r - r'}.$$

As we can diminish the denominator of this fraction as much as we please, without weakening the materials of the machine, there is no limit to the advantage of it, except what arises from the very great length of cord that must be used in raising W through a given space.

THE GENOU.

221. This instrument is represented in its simplest form in fig. 60, where AF is the profile of a frame in which the rods AB, BC work. AB is moveable about a fixed axis passing through A; it is connected with BC by a compass joint at B; and the other end C of BC, by means of a pin passing through it, is compelled to move in the vertical groove EF. The power is applied at G, a point in AB, in the plane of the rods ABC. It causes B to come nearer to AF; and, consequently, C presses downwards upon any obstacle opposed to it. It is obvious this machine is only applicable in those cases in which C is required to descend through a small space, as in printing, where it presses the paper upon the type.

Let W = the reaction at C, P the power applied horizontally at G, θ = the angle BAF, $a = AB$, $b = BC$, $c = AG$, and let GP intersect AF in p. Then $Gp = c \sin\theta$, and therefore the virtual velocity of P

$$= d\,(Gp) = c \cos\theta \,.\, d\theta.$$

Also $AF = a \cos\theta + b \cos BCA$, therefore the virtual velocity of W

$$= d\,(AF) = -a \sin\theta d\theta - b \sin BCA \,.\, d\,(BCA).$$

$$\text{Now } \sin BCA = \frac{a}{b} \sin \theta;$$

$$\text{and } \therefore \cos BCA \,.\, d(BCA) = \frac{a}{b} \cos \theta d\theta;$$

$\therefore$ the virtual velocity of W

$$= -a \sin \theta \,.\, d\theta - a \sin \theta \,.\, \frac{a}{b} \,.\, \frac{\cos \theta d\theta}{\cos BCA}$$

$$= -\left(1 + \frac{a}{b} \,.\, \frac{\cos \theta}{\cos BCA}\right) . \, a \sin \theta \,.\, d\theta.$$

Wherefore, by Art. 114, the advantage of the machine

$$= \frac{W}{P} = \frac{\cos \theta d\theta}{\left(1 + \frac{a}{b} \,.\, \frac{\cos \theta}{\cos BCA}\right) . \, a \sin \theta d\theta}$$

$$= \frac{c}{a} \,.\, \frac{b \cos BCA \,.\, a \cos \theta}{a \sin \theta \,(a \cos \theta + b \cos BCA)}$$

$$= \frac{AG \,.\, Cb \,.\, bA}{AB \,.\, Bb \,.\, AC},$$

where Bb is drawn parallel to GP.

222. A combination of wheels and axles may be used instead of the machine in Art. 220, when that is inconvenient and great advantage is required. Fig. 61 represents a combination of three of these mechanical powers. An endless strap passes over the axle a and the wheel B, and another strap passes over the axle b and the wheel C. If two successive wheels are required to turn in opposite directions, the strap must be crossed as between A and A in the figure; when the wheels are to turn in the same direction, the strap must not be crossed. B and C are turned by the friction of the straps upon their surfaces; and hence it is manifest, that if the force to be overcome by any wheel be greater than the friction of its strap, the strap will slip round without carrying the wheel with it, and the action of the machine will cease. Wherefore, in order to make the friction upon the surfaces of the wheels and axles as great as possible, they are covered with leather, which is nailed or glued on

them; and both this leather and the concave sides of the straps are suffered to be in a rough state; the friction is also increased by crossing the straps.

To calculate the advantage of this combination, denote the tension of the strings d and e by T, T'; then since P balances the tension T on the axle a, we have, by Art. 209,

$$\frac{T}{P} = \frac{\text{rad. of wheel } A}{\text{rad. of axle } a}.$$

Similarly, $$\frac{T'}{T} = \frac{\text{rad. of wheel } B}{\text{rad. of axle } b},$$

and $$\frac{W}{T'} = \frac{\text{rad. of wheel } C}{\text{rad. of axle } c};$$

and, therefore, by multiplying these equations together, we have

$$\frac{W}{P} = \frac{\text{product of radii of all the wheels}}{\text{product of radii of all the axles}}.$$

TOOTHED WHEELS.

223. By far the most general modification under which wheels and axles are used in practical Mechanics, is that of toothed wheels.

Let A, a (fig. 62) be the centres of two wheels BC, bc, upon the circumferences of which let teeth or cogs D, E, F, d, e, f, of any proposed form, be raised at equal distances all round; in order that this may be possible, the radii of the two wheels must be in proportion to the number of teeth that are to be constructed upon them. If one of the wheels (bc for instance) be turned round its axis a, its teeth will press upon the teeth of the other wheel BC, and turn it round its axis A in a contrary direction, and as two corresponding teeth F, f separate from each other in consequence of the motion, two others D, d come in contact; and thus the wheel a is enabled to produce a continuous motion in the wheel A. Similar teeth are constructed upon the axles of each wheel, and the axle so prepared is called a *pinion*, and its teeth are called *leaves*. From the nature of the wheel and axle it is manifest that motion is communicated to each wheel, in this modification, by a pinion in which it runs as in fig. 63, where P descending turns with it the pinion a which turns the wheel B, and this carries with it the the pinion b which turns the wheel C and axle c, and raises the weight W. In this case, as in Art. 222, it is clear that

$$\frac{W}{P} = \frac{\text{product of the radii of the wheels}}{\text{product of the radii of the axles}}$$

$$= \frac{\text{radius of } A}{\text{radius of } c} \times \frac{\text{product of number of teeth in the wheels}}{\text{product of number of leaves in the pinions}}.$$

Here there are no teeth in A and c, on which account we have not reduced their radii to equivalent numbers of teeth.

224. In the description of toothed wheels we have said that the teeth or cogs are to be of any proposed form, because in fact they are commonly made in any form that meets the fancy of the maker. It must not be imagined, however, that all forms are equally advantageous, as we shall easily understand by referring to fig. 62, and tracing the actions of the teeth upon each other during their motion. Suppose bc to begin to turn round, and let us trace the actions of d and D. When d first comes in contact with D, the latter presses against the side of d in a single line of points, very near the extremity of d, in the direction of a normal to the side of d, that is, in the direction pD perpendicular to the radius ad. Therefore, drawing Ap parallel to ad, the action of d may be transmitted to p, and its efficiency varies as Ap. But as the wheel bc continues turning, the point of contact D slides along the side of d, and thus produces a very strong friction, and consequently rapid wear both of the side of d and of the edge of the tooth D. This goes on till d and D come into the position e and E, when their sides are for a moment in contact, and then the efficiency of d in turning D varies as AD.

When the teeth d and D leave this position a similar action to what has just been described commences, only it is in a reverse order; and the edge of the tooth d presses against and rubs the side of the tooth D.

It appears then, with teeth of the form of those in this figure,—

1st. That the efficiency of the pressure which one tooth exerts upon another, and consequently the motion produced, is *very irregular*, being in one position proportional to Ap, and in another to AD.

2ndly. That the edges of the teeth are subject to very rapid wear in consequence of *rubbing* with a single line of points in contact with the sides of the teeth of the other wheel, which latter is thereby also very soon worn hollow, and the whole rendered useless.

3rdly. That in consequence of the rubbing of the teeth against each other much of the power is rendered ineffective.

4thly. That since there are favourable and unfavourable positions, the power must be sufficient to move the weight in the most unfavourable position with the requisite degree of celerity; and consequently, when the machine is in the most favourable position there will be an excess of power which will cause the machine to move much too rapidly, and often produce fractures; nothing in fact having so great a tendency to tear asunder the parts of a machine and render it useless as an irregular motion of this kind.

From these considerations it will at once be evident that the best form of the teeth will be, when,—

1st. The teeth of one wheel press upon those of the other in such a direction that the efficacy may be uniform; that is, such that the perpendiculars upon that direction from A and a are of constant lengths.

2ndly. The teeth of one wheel do not *rub* but *roll* upon those of the other.

3rdly. The motion of one tooth upon another is uniform.

When these conditions are fulfilled, it is also necessary that the distances of the axes of the wheels should be such that as great a number of teeth may be in contact at one time as possible, and that there may be no jolting nor violence of any kind when two teeth separate or come in contact. These precautions will very much diminish the chances of fracture.

Many forms of teeth have been proposed fulfilling one or more of those conditions, but it seems to be agreed on that the following is the best.

225. Let ABD (fig. 64) be a given wheel on which it is proposed to erect teeth; and let AB be the proposed breadth of a tooth. Upon AD wrap a string and fasten it at D. Then unwrap it, beginning at A, and its extremity A will trace out the curve Aa called the involute of the circle AD. In a similar manner, describe the involute Bb intersecting the former in C; then ACB will be the tooth required, which may be taken as a pattern of all the others to be formed upon the wheel. In a similar manner the leaves of the pinion may be found, by first constructing a pattern by means of the involute of its circumference. Let PL be a position of the thread whose extremity generates the involute Aa; then we may suppose the point L to be fixed for an instant, and therefore P will begin to describe an arc of a circle whose centre is L, and therefore PL is a normal to the curve AC, and OL the perpendicular upon this normal is constant. In the same manner it may be shewn, that the perpendiculars upon the normals to the leaves of the pinion are all constant and equal to the radius of the pinion. Wherefore, since the leaves of the pinion press against the teeth of the wheel in the directions of normals at the points of contact, and the perpendiculars on these directions are always of the same length, the action will be uniform, and consequently the motion will be uniform also.

THE ENDLESS SCREW.

226. This machine, represented in fig. 65, consists of a screw A whose axis is BC; and a wheel and axle D, E; the wheel being furnished with teeth exactly fitting the threads of the screw. The screw is turned by means of the winch CP, and its thread instead of pressing against a nut, presses against the teeth of the wheel, and forces them forward; each turn of the screw or winch, advancing the wheel one thread of the

screw; or, which is the same, one tooth of the wheel. The winch must therefore be turned round as many times as there are teeth in the wheel, in order to turn the axle E once round. Wherefore, putting R for the radius of the circle described by the power P; r for that of the axle E, and n for the number of teeth in the wheel D; the circumference described by P

$$= 2\pi R,$$

and therefore the space described in one turn of the wheel D

$$= 2n\pi R.$$

But the space ascended by W in the same time

$$\begin{aligned} &= \text{the circumference of the axle } E \\ &= 2\pi r. \end{aligned}$$

Consequently, by Art. 214,

$$\frac{W}{P} = \frac{2n\pi R}{2\pi r} = n \cdot \frac{R}{r}.$$

ON BALANCES.

227. A *balance* is any instrument invented for the purpose of comparing the heaviness of different bodies; that is, for ascertaining their *weights*.

The *common balance* (fig. 66) consists of an inflexible rod AB, called the beam, resting upon a fulcrum C at its middle point; from its extremities A, B are suspended two equal scales D, E by means of fine chains or strings. The fulcrum C and the points of support are in the same straight line, but the centre of gravity of the beam is a little below C. In this state the balance when unloaded ought to rest with its beam AB in a horizontal position. If a weight be put into one of the scales, the common centre of gravity of the scale and its load will be in the vertical passing through the point of support (Art. 131); and therefore we may transmit both the scale and its load to the point of support. Wherefore, when weights are placed in the scales, we may suppose them placed immediately at A and B, and therefore the balance becomes a straight lever whose fulcrum is C; and since the arms AC, BC are equal, there will be an equilibrium when the weights are equal (Art. 190). If the weights are unequal, let G (fig. 67) be the centre of gravity of the beam AB in the oblique position assumed in consequence of the inequality of the weights. Let w be the weight of the beam, which by Art. 130 we may suppose to be placed at G; S the weight of each of the equal scales; P, W the weights in D and E respectively; $\theta =$ the inclination of the beam to the horizon. Then the lever is kept at rest by three parallel forces, viz. $S+P$ at A, $S+W$ at B, and w at G. The perpendiculars from C upon the directions of these forces are

$$AC\cos\theta,\quad CB\cos\theta,\ \text{and}\ GC\sin\theta:$$

therefore, by Art. 194,

$$(S+P)\,.\,AC\cos\theta + w\,.\,GC\sin\theta = (S+W)\,.\,BC\cos\theta;$$

$$\therefore\ P\,.\,AC + w\,.\,GC\tan\theta = W\,.\,BC,$$

by dividing by $\cos\theta$, and observing that $AC = BC$;

$$\therefore\ \tan\theta = \frac{W-P}{w}\,.\,\frac{AC}{GC}.$$

The *sensibility* of a balance consists in the beam attaining considerable obliquity, when the difference between P and W is extremely small; and therefore the obliquity attained by different balances when loaded with the same weights, might be taken as a measure of their respective sensibilities. As $W-P$ is constant in this case, and as θ is very nearly equal to $\tan\theta$, we may use

$$\frac{AC}{w\,.\,GC}$$

as the measure of the sensibility.

A different measure of sensibility is however generally used, which may be thus explained. Let δ be the difference between W and P which produces a *given* very minute appreciable deviation θ' (which is the same for all balances);

$$\therefore\ \theta' \text{ or } \tan\theta' = \frac{\delta}{w}\,.\,\frac{AC}{GC};$$

$$\therefore\ \delta = w\,.\,\frac{GC}{AC}\,.\,\theta',$$

the ratio of the whole pressure $P + W + 2S + w$ (Art. 194) on the fulcrum to this weight is taken as the measure of the sensibility, or neglecting θ' in this measure which is the same for all balances, and using $2P + 2S + w$ for the pressure on the fulcrum, the fraction

$$\frac{P + S + \frac{1}{2}w}{w} \cdot \frac{AB}{GC}$$

is the measure generally employed. From either of these measures we derive the following general results:—

That the sensibility of a balance is increased,

(1) By increasing the length of the beam.

(2) By diminishing the distance of its centre of gravity from the fulcrum.

(3) By diminishing its weight.

For further information on subjects connected with the common balance, the reader is referred to Captain Kater's *Treatise on Machines*.

THE STEELYARD, OR ROMAN BALANCE.

228. This instrument is a lever AB (fig. 68) with unequal arms AC, CB; the fulcrum being C. As it is commonly constructed, the longer arm AC preponderates over the shorter CB; let therefore G be the centre of gravity of the beam AB, at which point we may suppose its weight w collected. And let P, a given weight suspended from p, balance W, the body to be weighed suspended from B. Then (Art. 194)

$$P \cdot Cp + w \cdot CG = W \cdot CB;$$

$$\therefore \quad W = \frac{P \cdot Cp + w \cdot CG}{CB};$$

$$\propto P \cdot Cp + w \cdot CG,$$

$$\propto Cp + \frac{w}{P} \cdot CG.$$

Now let D be such a point that when P is suspended from D, it just balances the beam;

$$\therefore\ P \cdot CD = w \cdot CG;$$

$$\therefore\ CD = \frac{w}{P} \cdot CG;$$

$$\therefore\ W \propto Cp + CD \propto Dp.$$

It appears therefore, that the weight W is proportional to the distance of p from D. If when p is at E, W is one pound, then making EF, FH, HI...each equal to DE; when p is at F, H, I... W will be 2 lbs., 3 lbs., 4 lbs.,... respectively, and we may number the points E, F, H... 1, 2, 3,... respectively; and if the spaces DE, EF... be subdivided into sixteen equal parts, each of them will correspond to one ounce, and we shall be able to ascertain W with corresponding accuracy by sliding the weight P along the arm AC until it comes into such a position as to balance W, and then reading off its place, which will be the number of pounds and ounces which express its weight.

The practical advantage of this balance is, that it requires but one weight P, and the pressure on the fulcrum, on which the friction depends, being equal to $P + W$, is less than the common balance so long as the substance to be weighed is heavier than P; on the contrary, however, when the substance to be weighed is not so heavy as P, the pressure on the fulcrum is greater than in the common balance, and consequently the friction, which diminishes the sensibility of the machine, is greater; and, therefore, for the determination of small weights the common balance is to be preferred, both on account of the diminution of friction and also because small weights can be more accurately subdivided than small spaces on the arm.

THE DANISH BALANCE.

229. This instrument consists of a lever AB (fig. 69), at one end A of which is fastened a given weight A, and at the other B a dish D to receive the substance to be weighed. The fulcrum or point of support C is made to slide along AB until the beam is horizontal, and by its place on the graduated beam AB the weight of the substance put into the scale-pan is determined. The method of graduating the beam AB may be thus

investigated. Let G be the centre of gravity of the instrument (including the beam, weight A, and scale-pan* D), P its weight; W the weight in the scale D. Then we may suppose P applied at G (Art. 133), and since there is an equilibrium between P and W about the fulcrum C,

$$\therefore\ W.BC = P.CG = P.(BG - BC)$$
$$= P.BG - P.BC;$$
$$\therefore\ BC = \frac{P.BG}{P+W}.$$

Wherefore, if P be $n$lbs. and W has the values 0, 1, 2, 3lbs.... BC has the values

$$\frac{n.BG}{n},\ \frac{n.BG}{n+1},\ \frac{n.BG}{n+2},\ \frac{n.BG}{n+3}\ldots$$

which quantities are in harmonical progression, because their reciprocals are in arithmetical progression. The spaces 0, 1; 1, 2; 2, 3; ... may be again subdivided, if necessary, and when this beam is thus prepared, the weight W may be ascertained with as much facility as in the common steelyard; but the disadvantage of this balance is, that as the weight increases the intervals between the divisions become smaller, and consequently it is not so well adapted for determining large weights as small ones.

ROBERVAL'S BALANCE.

230. This machine consists of four straight rods AB, Bb, ba, aA (fig. 70), forming a parallelogram in a vertical plane, and being connected by compass joints at B, b, a, A; at C and D the middle points of the rods AB and ab there are fixed axes about which they are moveable; GE, FH are two horizontal rods rigidly connected with Aa and Bb, from which the equal weights P and Q are suspended. The peculiarity of this balance is, that P and Q will be in equilibrium from whatever points of the rods GE and FH they are suspended. To prove this property, suppose the machine to be put in motion; then if

* The scale-pan is here supposed to be transmitted to B.

A ascends, B will descend through an equal space; and as $ABba$ must necessarily continue to be a parallelogram, Aa and Bb will continue parallel to CD, and therefore each vertical; wherefore E will ascend and F will descend through spaces respectively equal to those described by A and B, and therefore equal to each other. It is also manifest, since Aa and Bb continue vertical during the motion, that GE and FH continue horizontal, and consequently the space ascended by P is equal to that descended by Q, wherefore they satisfy the equation of Art. 217, and are consequently in equilibrium in every position.

CHAPTER VIII.

ON FRICTION.

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231. THE resistance to rotatory and progressive motion in bodies which rub against surfaces with which they are in contact, is called *friction*, and is distinguishable into two kinds.

(1) *Statical* friction, or resistance to the *production* of motion in a quiescent body.

(2) *Dynamical* friction, or the resistance which diminishes *existing* motion.

Of these two kinds, since all machines are designed to work, the latter is of more importance in practical Mechanics.

232. There are three ways in which one surface can move upon another, and hence both Statical and Dynamical friction are divided into three corresponding heads.

(1) When the surfaces in contact are two planes.

(2) When the surfaces in contact are a solid and a hollow cylinder.

(3) When a cylinder rolls (without rubbing) upon a plane.

The laws which govern the action of friction cannot be deduced from theoretical considerations, though theory will render us great assistance in our researches by pointing out the experiments which are most likely to lead us to the discovery of them, as well as shewing the inconclusiveness of other experiments, on which we might otherwise be induced to rely. It is to be regretted, however, that the experiments which have been made upon the subject by different philosophers are frequently at variance; and, consequently, the theory cannot be said to have arrived at that state of perfection which is desirable.

233. *The statical friction of plane surfaces is, under like circumstances, proportional to the pressure.*

For let AB, ab be two planes in contact, placed in a horizontal position, the lower one AB being firmly fixed, but the upper one ab free to slide upon it. To ab attach a horizontal string bD passing over a pulley D, and having a dish C suspended from it. Load ab with a weight w, and denote the whole pressure of the plane ab on AB by W. Pour fine sand into the dish C until it begins to move, and then the weight of the dish and sand is the measure of the statical friction of the planes corresponding to the pressure W. If ab be loaded with more weights until the pressure is $2W$, the friction is found to be double of what it was before; when the pressure is $3W$, the friction is trebled; and so on. Wherefore the statical friction of plane surfaces is proportional to the pressure.

This result was confirmed by Coulomb and Ximines for very considerable pressures; in extreme cases, where the pressures were very large indeed, the friction was observed to be rather less in proportion than for small pressures; the deviation from the above law was however so small, even for extreme cases, that we shall not fall into any very considerable error in supposing the law to be universally true.

The following method of establishing the property of the proportionality of the friction to the pressure, is very convenient for experiments.

Let the body W (fig. 51) be placed upon an inclined plane AB, and then let the altitude BC be slowly increased until the plane has acquired such an elevation that W begins to slide down it; at this moment the friction just balances the weight W, and since it acts parallel to the plane in the direction AB, we may consider W as kept in equilibrium by a power in that direction, hence

$$\frac{\text{friction}}{W} = \sin i, \quad (\text{Art. } 213)$$

$$\frac{W}{\text{pressure}} = \frac{1}{\cos i}, \quad (\text{Art. } 214)$$

$$\therefore \frac{\text{friction}}{\text{pressure}} = \frac{\sin i}{\cos i} = \tan i;$$

$$\therefore \text{friction} = (\text{pressure}) \,.\, \tan i.$$

234. The fraction $\frac{\text{friction}}{\text{pressure}}$, is usually called the *coefficient of friction*, and is taken as its measure. It appears then, that in the last experiment the coefficient of friction is equal to the tangent of the inclination of the plane.

235. It being granted that the friction is proportional to the pressure when the surfaces are given, then, whatever be the magnitude of the surfaces in contact, the friction will remain the same, so long as the pressure is the same.

Let the body W (fig. 51) have faces, whose areas are C and D square inches; then when the first face is in contact with the plane, the whole pressure is supported on C square inches, and therefore the pressure on each square inch, is equal to

$$\frac{\text{pressure}}{C};$$

and therefore the friction upon each square inch of surface

$$= \frac{\text{pressure}}{C} \,.\, \tan i.$$

Consequently the friction upon the whole surface

$$= \frac{\text{pressure}}{C} \,.\, \tan i \times \text{number of square inches}$$

$$= \frac{\text{pressure}}{C} \,.\, \tan i \times C$$

$$= (\text{pressure}) \,.\, \tan i.$$

In the same way it may be shewn that the friction upon the second surface

$$= (\text{pressure}) \,.\, \tan i,$$

and therefore the friction of a body is the same whether the surface on which it rests be large or small. When the surface is very small in proportion to the weight, the pressure on each square inch becomes very large, and then the friction, as observed in Art. 233, becomes somewhat less in proportion to the pressure; and therefore the friction is less, in a slight degree, when the body rests upon a small surface than a larger.

236. These are the chief properties of statical friction; it does not belong to us to investigate those of dynamical friction; but to make the subject complete we shall annex the following summary of results which have been obtained by various experimentalists.

(1) Dynamical friction is a uniformly retarding force: and it diminishes as the pressure increases.

This is only true when the surfaces in contact are hard; for in experiments made with bodies covered with cloth, woollen, &c. the friction was found to increase with the velocity.

(2) In the same body Statical friction is greater than Dynamical friction; i.e. it requires a greater force to put a body at rest in motion, than is requisite to preserve the motion undiminished when once it is produced.

This was thought by Professor Vince to arise from the cohesion of the body to the plane when it is at rest, which does not happen when the body is in motion.

(3) When a body of wood is first laid upon another, the Statical friction increases for a few minutes, when it attains its maximum, and no further alteration takes place. In making experiments, therefore, it is necessary to wait some time before the body is put in motion.

(4) Friction between substances of the same kind is greater than when they are of different kinds.

(5) The velocity has very little, if any, influence except when one body is composed of wood and the other of metal, in which case the resistance increases with the velocity.

(6) It is also found that friction is diminished by oiling and polishing the surfaces in contact. There is a limit however to the latter, for if they be very highly polished, the resistance increases.

(7) The friction of cylinders rolling on planes, is proportional to their pressures directly and their radii inversely.

It is remarkable, that friction of this kind, unlike that between two planes, is not diminished by greasing or oiling the surface of the planes and cylinder. This kind of friction is much less than that produced by rubbing.

CHAPTER IX.

ON ELASTIC STRINGS.

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237. STRINGS made of certain substances are found to be elastic; that is, they admit of being lengthened by the application of forces to their extremities, and regain their original dimensions, or nearly so, when the forces are removed. Spiral springs composed of steel wire, such as the one exhibited in fig. 71, are found to possess the same property in a remarkable degree. The connection between the force which stretches a string, or a spring of the kind here mentioned, and the increase of length cannot be investigated from mathematical considerations, but is to be determined entirely by experiments.

Let MN (fig. 72) be a very smooth horizontal table; AB an elastic string or spring laid upon it and fastened at A; W a weight stretching the string by means of a thread passing over the pulley C, whose position is such that ABC coincides with the table. Then, if W stretches the string to b, and another weight W' stretches it still farther to b', it is found that

$$Bb : Bb' :: W : W';$$

that is, *the excess of a given elastic string or spiral spring above its natural length is proportional to the weight which stretches it.*

238. *This excess is, in different strings of the same make and materials, proportional to their lengths.*

For the tension of a string being the same in every part, if we divide the string into any number of equal parts, the increase of length in each part will be the same, and therefore the increase of the whole string will be proportional to the number of these equal parts which it contains: that is, to its length.

239. Consequently, upon the whole, the increase of length of a string is proportional to

(its length) × (weight which stretches it).

Wherefore if L be the natural length of a string, and l its length when stretched by a weight W,

$$l - L \propto L \,.\, W = C \,.\, LW;$$

where C denotes a constant dependent on the material, thickness and make of the string.

240. *Suppose the string* AB (fig. 73), *whose natural length is* a, *to be suspended vertically from one end* A, *and stretched by its own weight* w *only; to determine the increase of its length.*

In AB take any points P, Q very near to each other, and when the string is stretched let b, p, q, a be the points corresponding to B, P, Q, A; $x = BP$, $\delta x = PQ$, $y = bp$, $\delta y = pq$. Then δx is stretched into δy by the weight of bp or BP which $= \dfrac{wx}{a}$;

$$\therefore\ \delta y - \delta x = C \,.\, \delta x \,.\, \frac{wx}{a};$$

therefore, dividing by δx, and taking the limits,

$$d_x y - 1 = C \,.\, \frac{wx}{a};$$

$$\therefore\ y - x = \frac{C}{2} \,.\, \frac{wx^2}{a}, \text{ by integration};$$

$$\therefore\ ab - AB = \frac{C}{2} \,.\, \frac{wa^2}{a} = \tfrac{1}{2} Cwa.$$

Hence *the increase is one half of what it would be, if* AB *were stretched upon a horizontal table by a weight equal to its own weight.*

241. *A weight* W *is now suspended from* b, *to determine the further increase of length.*

The weight which stretches pq is, in that case,

$$W + \frac{wx}{a};$$

$$\therefore\ \delta y - \delta x = C.\delta x.\left(W + \frac{wx}{a}\right);$$

$$\therefore\ d_x y - 1 = C\left(W + \frac{wx}{a}\right);$$

$$\therefore\ y - x = C\left(Wx + \frac{wx^2}{2a}\right);$$

$$\therefore\ ab - AB = CWa + \tfrac{1}{2}Cwa.$$

242. Of this increase the part $\frac{1}{2}Cwa$ we have seen is due to the weight of the string, and therefore CWa, the part due to the weight W, is the same as if the string had no weight. Hence *when a string is stretched by several forces, each one produces the same increase of length as it would do if the other forces did not act.*

By way of illustration we shall add the following examples.

243. *Two weights* P, Q (fig. 74) *resting on two inclined planes* AB, AC, *are connected by a given elastic string; to find the position of equilibrium.*

Let α, β be the inclinations of AB, AC, and θ that of PQ to the horizon; a = the natural length of PQ; T = its tension. Then P is kept in equilibrium on the plane AB by the force T acting in the direction PQ;

$$\therefore\ T\cos APQ = P\sin\alpha, \text{ (Art. 212).}$$

But $APQ = \alpha - \theta$;

$$\therefore\ T = \frac{P\sin\alpha}{\cos(\alpha-\theta)}.$$

$$\text{Similarly,}\quad T = \frac{Q\sin\beta}{\cos(\beta+\theta)};$$

$$\therefore\ P\frac{\cos(\beta+\theta)}{\cos\theta\sin\beta} = Q\frac{\cos(\alpha-\theta)}{\cos\theta\sin\alpha};$$

$$\therefore\ P(\cot\beta - \tan\theta) = Q(\cot\alpha + \tan\theta);$$

$$\therefore\ \tan\theta = \frac{P\cot\beta - Q\cot\alpha}{P+Q}, \text{ which gives } \theta;$$

$$\text{and } PQ = a + C.a.T$$

$$= a\left\{1 + \frac{C.P\sin\alpha}{\cos(\alpha-\theta)}\right\}.$$

From which PQ is known and thence AP and AQ by means of the triangle APQ, whose angles are all known.

244. *Two equal weights* P, Q (fig. 75) *are connected by an elastic string, whose natural length is* BC; *to find the nature of the curves* BP, CQ, *on which they will always rest in equilibrium with the string parallel to the horizon; the plane of the curves being vertical.*

It is manifest, since the weights are equal, that the curves must also be equal. Bisect BC in A, and draw AM vertical; $AB = AC = a$, $AM = x$, $MP = MQ = y$, $T =$ the tension of PQ;

$$\therefore PQ - BC = C \,.\, BC \,.\, T,$$

$$\text{or } 2y - 2a = C \,.\, 2a \,.\, T;$$

$$\therefore y - a = CaT.$$

But P being sustained upon the curve BP by its gravity P and the force T, we have by Art. 213,

$$T = Pd_y x;$$

$$\therefore y - a = CaPd_y x;$$

$$\therefore (y - a)^2 = 2CaPx, \text{ by integration,}$$

which is the equation of a parabola. Hence BP, CQ are two semi-parabolas, whose vertices are B, C.

CHAPTER X.

ON THE FUNICULAR POLYGON, ON THE CATENARY, ON ROOFS AND BRIDGES.

ON THE FUNICULAR POLYGON.

245. ABCDEF (fig. 76) *is a cord, supposed devoid of weight, suspended from two points* A, F *in a horizontal line; at the knots* B, C, D, E...... *weights*, W_1, W_2, W_3, W_4...... *are hung; to determine the proportions of these weights that it may hang in a given form.*

From A draw Ac, Ad, Ae, Af respectively parallel to the portions BC, CD, DE, EF of the cord; and denote the respective inclinations of AB, BC, CD......to the horizontal line AF by α, β, γ......; draw MB vertical. Then B is kept at rest by the tensions of AB, BC and the weight W_1, which forces are respectively parallel to the sides BA, Ac, cB of the triangle ABc, and are therefore proportional to them. Therefore W_1 is proportional to Bc. In the same manner W_2 is proportional to cd; and they are on the same scale, for in both Ac represents the tension of BC.

$$\therefore \frac{W_1}{W_2} = \frac{Bc}{cd} = \frac{BM - cM}{cM - dM}$$

$$= \frac{AM \tan \alpha - AM \tan \beta}{AM \tan \beta - AM \tan \gamma}$$

$$= \frac{\tan \alpha - \tan \beta}{\tan \beta - \tan \gamma}.$$

Similarly, $$\frac{W_2}{W_4} = \frac{\tan \beta - \tan \gamma}{\tan \gamma - \tan \delta}\ldots\ldots$$

It appears, therefore, that any one of the weights is proportional to the difference of the tangents of the angles at which the two sides of the polygon, which form the angle at which it is suspended, are inclined to the horizon.

The angles MAe, Maf, which are situated above the line AF, are to be accounted negative.

246. The horizontal tension of any part of the string is represented by AM, for it is the resolved part of the lines AB, Ac, Ad...... which represent the whole tensions; and this horizontal tension : any weight (W_2 suppose)

$$:: AM : cd :: 1 : \tan\beta - \tan\gamma.$$

COR. The tension of any string BC : the horizontal tension

$$:: Ac : AM :: AM \sec\beta : AM :: \sec\beta : 1.$$

247. If AB, BC, CD...... in the preceding figure, instead of being lines devoid of weight, be heavy beams of wood, or bars of metal, connected at the joints A, B, C, D...... by hinges, we must consider each beam as exerting by means of its weight vertical forces at its extremities. Thus, if w_1, w_2, w_3...... be the weights of AB, BC, CD......we may consider BC as exerting equal pressures $\frac{1}{2}w_2$ at B and C in a vertical direction, the centre of gravity of the beam being supposed at its middle point; in like manner AB exerts a vertical pressure equal to $\frac{1}{2}w_1$ at B, and therefore we may consider $W_1 + \frac{1}{2}(w_1 + w_2)$ as the whole weight suspended at B. Similarly, the weights to be considered as suspended at C, D......are respectively

$$W_3 + \tfrac{1}{2}(w_2 + w_3)\,; \quad W_3 + \tfrac{1}{2}(w_3 + w_4)\,;\ldots\ldots$$

and these weights are to be used instead of those given in the preceding articles.

These considerations are intimately connected with the construction of suspension bridges.

248. If W_1, W_2, W_3......are evanescent, then the weights to be considered as suspended are $\frac{1}{2}(w_1 + w_2)$, $\frac{1}{2}(w_2 + w_3)$...... and if the beams are all equal, each of these become equal to w_1.

ON ROOFS AND BRIDGES.

249. If the whole figure of Art. 245, be inverted or turned round the horizontal line AF through an angle of 180^0, as in fig. 77, we shall find the same relations between the weights as before; it will also appear, from the same reasoning as in Art. 247, that the weights to be considered as hanging from B, C, D...... are the same as there investigated. In this state the problem contains the whole theory of roofs, arches, and bridges. If $ABCDEF$ be considered as a roof, of which AB, BC...... are the beams, then the horizontal thrust at A and F tending to push out the walls on which the roof is erected, is represented by AM, on the same scale as that wherein Bc represents the weight to be suspended from B; it is therefore equal to

$$\frac{W_1 + \frac{1}{2}(w_1 + w_2)}{\tan\alpha - \tan\beta}.$$

This thrust is usually prevented from taking effect upon the walls by inserting the ends, A, F of the beams AB, FE into another AF called the *tie-beam*, which is thus made to sustain the whole thrust; at other times the walls are prevented from bulging by buttresses, or shores, built against them.

If it were required to construct a roof of given span with given beams, which has to support given weights, we must take an equal number of smaller proportional beams, and connect them by strings or pins at the joints, so as to allow them to move freely, and load them with proportional weights. Then if this model be suspended from its extremities at a proportional distance, as in Art. 245, it will assume the required form, which we have merely to turn round AF through an angle of 180^0, and it will be a perfect model of the required roof; and will possess the property of being in equilibrium in every part. In such a roof there will be no unnecessary strain on any part of the materials of which it is constructed, and consequently no part will require to be unnecessarily strong. In this simple manner we may also obtain the model of a bridge of given span, by taking a great number of very short beams to represent the arch stones, and connecting them as before. If when we suspend this model-string of arch stones loaded with weights proportional to what (in the place they occupy in the bridge) they will have to sustain, we find that the bridge would be too lofty, we must remove the points of suspension farther apart, until we have obtained the proper altitude. This method will give us a bridge, in perfect equilibrium in every part, and in which there is, therefore, no injurious strain, no useless strength, nor dangerous weakness in any part.

ON THE CATENARY.

A *Catenary* is the curve assumed by a fine chain or flexible string when suspended from its extremities.

250. *To investigate the equation of the catenary.*

Let AOF (fig. 78) be the catenary; A, F being the points from which the chain is suspended, and O being the lowest point of it. Through O draw BOD vertical, which take for the axis of x, O being the origin. From P any point of the chain draw PM perpendicular to OD; and draw PT a tangent at P. $x = OM$, $y = MP$, $s = OP$. Since there is equilibrium we may suppose the part OP to become rigid; then since it is kept in equilibrium by the action of three forces (its own weight and the tensions at P and O), which act upon it in the directions of the sides of the triangle MTP taken in order, we have

$$\frac{\text{tension at } O}{\text{weight of } OP} = \frac{PM}{MT} = \tan PTM = d_x y.$$

But if the chain be uniform, the weight of OP may be represented by its length s, and the tension at O by the length of a piece of the same chain of the length a;

$$\therefore\ \frac{a}{s} = d_x y;$$

$$\therefore\ \frac{a^2}{s^2} + 1 = (d_x y)^2 + 1 = (d_x s)^2;$$

$$\therefore\ 1 = \frac{s d_x s}{\sqrt{a^2 + s^2}};$$

$$\therefore\ x + C = \sqrt{a^2 + s^2}.$$

But when $x=0$, $s=0$, and therefore $C=a$;

$$\therefore\ x+a=\sqrt{a^2+s^2},$$

$$\text{and }\therefore\ x^2+2ax=s^2,$$

which is the relation between any arc and its abscissa.

251. *To find the equation of the catenary in terms of the rectangular co-ordinates* x, y.

The equation required is expressed in its most simple form by taking for the origin of co-ordinates the point B, which is such that $OB=a$. Let then $BM=x$; then from the last article,

$$\sqrt{a^2+s^2}=x;$$

$$\therefore\ \sqrt{x^2-a^2}=s;$$

$$\therefore\ \frac{x}{\sqrt{x^2-a^2}}=d_x s$$

$$=\sqrt{1+(d_x y)^2};$$

$$\therefore\ d_x y=\frac{a}{\sqrt{x^2-a^2}};$$

$$\therefore\ y=a\log_e\left(x+\sqrt{x^2-a^2}\right)+C.$$

But when $x=a$, $y=0$, consequently $C=-a\log_e a$;

$$\therefore\ y=a\log_e\frac{x+\sqrt{x^2-a^2}}{a}.$$

This is the equation required.

252. The relation between x and y, and that between s and y may be expressed in very simple exponential forms as follows.

From the last equation we have

$$\frac{x}{a}+\left(\frac{x^2}{a^2}-1\right)^{\frac{1}{2}}=e^{\frac{y}{a}};$$

$$\therefore\ \frac{x}{a}-\left(\frac{x^2}{a^2}-1\right)^{\frac{1}{2}}=e^{-\frac{y}{a}};$$

$$\therefore \quad \frac{2x}{a} = e^{\frac{y}{a}} + e^{-\frac{y}{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{Again,} \quad \frac{2s}{a} = 2\sqrt{\frac{x^2}{a^2} - 1}$$

$$= e^{\frac{y}{a}} - e^{-\frac{y}{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

253. Def. If through B we draw BC horizontal, it is called the *directrix* of the catenary.

254. *The tension of the chain at any point* P *is equal to the weight of a piece of the same chain of the length* BM.

For $$\frac{\text{tension at } P}{\text{tension at } O} = \frac{TP}{PM} = \frac{1}{\sin PTM}$$

$$= \frac{\sqrt{1 + (d_x y)^2}}{d_x y}$$

$$= \frac{\sqrt{s^2 + a^2}}{a}$$

$$= \frac{x}{a}$$

$$= \frac{\text{weight of length } x}{\text{weight of length } a}.$$

But the denominators of these fractions are equal;

$\therefore$ tension at P = weight of chain of length x.

255. We have supposed the chain to be uniform; if it should be of variable density or of variable thickness, let ρ be such a quantity that $\rho\delta s$ may represent the mass of a small element $(\delta s =)$ PQ of the chain. Then the weight of OP is $\Sigma\,(g\rho\delta s) = g\int\rho ds = g\int_y \rho d_y s$; and representing the tension at O by ga, we have by proceeding as in Art. 250,

$$d_x y = \frac{\text{tension at } O}{\text{weight of } OP}$$

$$= \frac{ga}{g\int_y \rho d_y s};$$

$$\therefore \int_y \rho d_y s = \frac{a}{d_x y} = a d_y x;$$

$$\therefore \rho d_y s = a d_y^2 x \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

When ρ is given, this equation being integrated will give the form of the catenary.

256. *To find the law of density that the catenary may be of a given form.*

In this case the relation between x and y is given to find ρ. From the last Article we have

$$\rho \cdot \frac{(d_y s)^3}{d_y^2 x} = a\,(d_y s)^2.$$

Now the quantity which is multiplied into ρ is the radius of curvature of the curve at P, and $d_y s$ is the secant of the inclination of the tangent at P to the horizon; wherefore

$$\rho = \frac{a \cdot \sec^2 \text{ of the inclination}}{\text{radius of curvature}}.$$

257. *To find the form of the catenary when the chain is acted on by a force tending to a fixed centre.*

Let BAC (fig. 79) be the catenary, suspended from B, C. S the centre of force, A that point of the chain which is nearest to S; therefore SA is a normal at A. Let P be any point, and PQ a small element of the curve. $a = SA$, $s = AP$, $\delta s = PQ$, $r = SP$, $r + \delta r = SQ$, $t =$ tension of the chain at P, $t + \delta t =$ the tension at Q, $\rho\delta s =$ the mass of PQ, and $F =$ the force which acts at P towards S. Then the weight of $PQ = F\rho\delta s$, which we may suppose to act ultimately in the direction QS. Hence resolving the forces, which act upon PQ, in the direction of the tangent at Q or P, we have

$$t + F\rho\delta s \cos PQS = t + \delta t.$$

But if we draw PP' perpendicular to SQ, we have

$$\delta s \cos PQS = \delta r;$$

$$\therefore F\rho\delta r = \delta t;$$

$$\therefore \int F\rho dr = t \dots\dots (1).$$

The integral is to be taken from $r = SA$ to $r = SP$.

Again, the chain AP is kept in equilibrium by the tensions at A and P, and by the weight of each particle of it in a direction passing through S. Hence taking the moments of all these forces about S, we have

$$a \, . \, (\text{tension at } A) = pt,$$

where p is the perpendicular from S upon the tangent at P.

The left hand member of this equation is constant, and therefore representing it by C, we have

$$t = \frac{C}{p} \quad \text{..................... (2).}$$

Hence, combining equations (1) and (2), we have

$$\int F\rho dr = \frac{C}{p};$$

$$\therefore \; F\rho = -\frac{Cd_r p}{p^2}.$$

When ρ is given, this equation being integrated will give the form of the catenary.

258. *To find the law of density that the chain may hang in a given form when acted on by a given central force.*

In this case the relations between F, r and p are given to find ρ. From the last Article we have

$$\rho = -\frac{Cd_r p}{Fp^2}.$$

259. Cor. Since $QP' = \delta r$, $F\rho\delta r =$ the weight of a piece of the given chain of the length QP' and density ρ; hence if the density of the chain be the same throughout, the equation (1) taken between its proper limits gives

tension at P – tension at A

= weight of chain of the length PS

– weight of chain of the length SA.

This result corresponds to that obtained in Art. 254.

260. *To find the form of the catenary when the chain is acted on by any forces in its own plane.*

Let AB (fig. 80) be the curve, in the plane of which take two lines Ox, Oy perpendicular to each other for co-ordinate axes. Let X, Y be the components of the accelerating force which acts at P, parallel to Ox, Oy respectively. Let PQ be a very small arc; PT, QT tangents at P and Q meeting in T. From T draw TG a normal to the curve.

Let $x = OM$, $y = MP$; $s = AP$, $\delta s = PQ$; $x + \delta x$, $y + \delta y$ the co-ordinates of Q; and $\rho\delta s$ the mass of the arc PQ. We suppose δs so small that the accelerating forces X, Y may be considered the same for every point of it; consequently $X\rho\delta s$, $Y\rho\delta s$ are the weights of PQ estimated parallel to Ox, Oy respectively. Now PQ is kept at rest by three forces, the tension t at P, the tension $t + \delta t$ at Q, and the resultant of $X\rho\delta s$, $Y\rho\delta s$; consequently, as PQ may be considered rigid without disturbing the equilibrium, these three forces all pass through the point T; they therefore satisfy the conditions of equilibrium of forces acting on a point. Resolve the forces parallel and perpendicular to the normal GT;

$$\therefore\ 0 = t\cos PTG + (t + \delta t)\cos QTG - X\rho\delta s \,.\, d_s y + Y\rho\delta s \,.\, d_s x,$$

and

$$0 = t\sin PTG - (t + \delta t)\sin QTG - X\rho\delta s \,.\, d_s x - Y\rho\delta s \,.\, d_s y.$$

$$\text{Now } \cos PTG = \frac{\tfrac{1}{2}\delta s}{\text{rad. curv.}} = \tfrac{1}{2}\delta s\sqrt{(d_s^2 x)^2 + (d_s^2 y)^2};$$

$$\text{and } \sin PTG = 1 \text{ ultimately};$$

hence by substitution and dividing by δs, we have

$$0 = t\sqrt{(d_s^2 x)^2 + (d_s^2 y)^2} - X\rho d_s y + Y\rho d_s x,$$

$$\text{and } 0 = d_s t + X\rho d_s x + Y\rho d_s y.$$

By eliminating t between these equations, we obtain the differential equation of the required curve.

[The remaining Articles of this Chapter are from the pen of the Rev. J. A. Coombe, Fellow of St John's College: by whose permission they are here inserted.]

261. PROP. *To find the form of equilibrium of a uniform inextensible string on a surface and acted on by any forces.*

Let $u=0$ be the equation to the surface, xyz the rectangular co-ordinates of any point in the string, and therefore of a point in the surface; s the length of a portion of the string intercepted between a fixed point in the string and the point (xyz); XYZ the resolved parts of the forces at the point (xyz) parallel to x, y, z: R the normal reaction at the point (xyz), making angles $\alpha\beta\gamma$ with the axis of co-ordinates; T the tension of the string at the point (xyz), one extremity of an element δs of the string, and acting in the tangent at that point.

Hence Td_sx will be the resolved part in x, and

$$Td_sx + d_s(Td_sx) \,.\, \delta s$$

will ultimately be the resolved part in x of the tension at the other extremity.

Hence $d_s(Td_sx)\,\delta s$ will be the difference of the resolved parts in x.

The other forces acting on δs parallel to x are $X\delta s$ and $R\delta s\cos\alpha$, supposing the forces to act equally on every point of the very small element δs.

Now δs being at rest under the action of these forces, we may suppose it to become rigid and apply the equations of equilibrium. Hence we have (dividing by δs),

$$d_s(Td_sx) + X + R\cos\alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Similarly, $$d_s(Td_sy) + Y + R\cos\beta = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and $$d_s(Td_sz) + Z + R\cos\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The equations of moments may be dispensed with for the following reasons. Consider three adjacent points P, Q, R, in the curve, Q being in the middle, and the tangents PT, RT, meeting in T. Then the plane containing these points and the tangents PT, RT will be the osculating plane.

Now the forces X and $R\cos\alpha$ being supposed to act equally on every point of PR will have a resultant through Q, and so will the like forces parallel to y and z.

Hence the whole of the forces acting on PR being reducible to three, lying in the osculating plane, will pass through a single point T in that plane; and the equations of moments taken about this point will be identical.

Recurring to the above equations, our object will be to eliminate T and R between them.

Now $(1).\,d_s x + (2).\,d_s y + (3).\,d_s z = 0$ gives

$$d_s T + X d_s x + Y d_s y + Z d_s z + R\,(\cos\alpha d_s x + \cos\beta d_s y + \cos\gamma d_s z) = 0,$$

since $(d_s x)^2 + (d_s y)^2 + (d_s z)^2 = 1$;

and $\therefore\ d_s x \,.\, d_s^2 x + d_s y \,.\, d_s^2 y + d_s z \,.\, d_s^2 z = 0.$

Also because the tangent to the curve is perpendicular to the normal to the surface, we have

$$\cos\alpha \,.\, d_s x + \cos\beta \,.\, d_s y + \cos\gamma \,.\, d_s z = 0.$$

Hence the above equation becomes

$$d_s T + X d_s x + Y d_s y + Z d_s z = 0;$$

or if $X d_s x + Y d_s y + Z d_s z = d_s v$,

we have $T + v = 0$ ....(4);

(the arbitrary constant being included in v).

Again, $(1).\,d_s y - (2).\,d_s x = 0$ gives

$$T\,(d_s y d_s^2 x - d_s x d_s^2 y) + X d_s y - Y d_s x + R\,\{\cos\alpha d_s y - \cos\beta d_s x\} = 0.$$

Now if $\dfrac{1}{V} = \{(d_x u)^2 + (d_y u)^2 + (d_z u)^2\}^{\frac{1}{2}}$,

we have $\cos\alpha = V d_x u$, $\cos\beta = V d_y u$, $\cos\gamma = V d_z u$,

the differential coefficients of u being partial.

Hence the above equation becomes

$$T\,(d_s y d_s^2 x - d_s x d_s^2 y) + X d_s y - Y d_s x + R V\,(d_x u d_s y - d_y u d_s x) = 0 \text{......(5)}.$$

Similarly,

$$T(d_s x d_s^2 z - d_s z d_s^2 x) + Z d_s x - X d_s z + RV(d_s u d_s x - d_x u d_s z) = 0 \quad \text{......(6)},$$

and

$$T(d_s z d_s^2 y - d_s y d_s^2 z) + Y d_s z - Z d_s y + RV(d_y u d_s z - d_z u d_s y) = 0 \quad \text{......(7)}.$$

Hence $(5).\,d_z u + (6).\,d_y u + (7).\,d_x u = 0$ gives on substituting for T its value derived from (4),

$$\left.\begin{aligned} &v d_z u\,(d_s y d_s^2 x - d_s x d_s^2 y) \\ +\,&v d_y u\,(d_s x d_s^2 z - d_s z d_s^2 x) \\ +\,&v d_x u\,(d_s z d_s^2 y - d_s y d_s^2 z) \end{aligned}\right\} = \left\{\begin{aligned} &d_z u\,(Y d_s x - X d_s y) \\ +\,&d_y u\,(X d_s z - Z d_s x) \\ +\,&d_x u\,(Z d_s y - Y d_s z) \end{aligned}\right\} \quad \text{............}(A).$$

This equation together with $u = 0$ are the equations to the curve of double curvature into which the string is arranged.

262. Cor. 1. Suppose the resultant K of the forces XYZ, acting at the point (xyz), is in the normal to the surface at that point, so that

$$X = KVd_x u, \quad Y = KVd_y u, \quad Z = KVd_z u,$$

the equation (A) then takes the form

$$\left.\begin{aligned} &d_z u\,(d_s y d_s^2 x - d_s x d_s^2 y) \\ +\,&d_y u\,(d_s x d_s^2 z - d_s z d_s^2 x) \\ +\,&d_x u\,(d_s z d_s^2 y - d_s y d_s^2 z) \end{aligned}\right\} = 0 \quad \text{..............} \quad (8),$$

or substituting A, B, C for the coefficients of $d_x u$, $d_y u$, $d_z u$, the equation is

$$Ad_x u + Bd_y u + Cd_z u = 0.$$

Now the equation to the osculating plane at the point (xyz) is

$$A(x' - x) + B(y' - y) + C(z' - z) = 0 \quad \text{...........} \quad (9),$$

and the equations to the normal are

$$\frac{d_x u}{x' - x} = \frac{d_y u}{y' - y} = \frac{d_z u}{z' - z} \text{.................}(10),$$

$x'y'z'$ being the current co-ordinates of a point in the plane or normal: and when the plane (9) contains the normal (10) we have the condition

$$Ad_x u + Bd_y u + Cd_z u = 0.$$

Hence equation (8) expresses that the osculating plane contains the normal: now this is the property of *the shortest line* between two points on the surface.

263. PROP. *To find the pressure on the surface at any point.*

$$(1).\ \mathrm{Cos}\,\alpha + (2).\cos\beta + (3).\cos\gamma = 0,$$

gives
$$-R = X\cos\alpha + Y\cos\beta + Z\cos\gamma + T\{d_s^2x\cos\alpha + d_s^2y\,.\cos\beta + d_s^2z\cos\gamma\}.$$

Now if ρ be the radius of absolute curvature at the point (xyz), and λ, μ, ν the angles it makes with the axes, we have

$$\cos\lambda = \rho d_s^2x,\quad \cos\mu = \rho d_s^2y,\quad \cos\nu = \rho d_s^2z;$$

$$\therefore\ -R\delta s = X\delta s\cos\alpha + Y\delta s\cos\beta + Z\delta s\cos\gamma + T.\frac{\delta s}{\rho}(\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu).$$

Let θ be the angle between the radius of absolute curvature, in the direction of which the resultant of the tensions on the extremities of δs acts, and the normal to the surface, then

$$\cos\theta = \cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu.$$

Substituting in the above equation, we have then

pressure on a portion δs of the surface

= resolved force in the normal

+ resolved tension in the normal.

264. COR. When there are no forces acting on the string, so that $X=0$, $Y=0$, $Z=0$, we have

$$d_sT = 0,$$

$$\text{or}\ \ T = \text{constant} = k,$$

$$\text{and pressure} = \frac{k}{\rho}\ \text{on an unit of length},$$

$$\propto \frac{1}{\rho}.$$

265. PROP. *To find the form of equilibrium when the string is not attached to a surface.*

The equations (1), (2), (3), will give the equations of equilibrium, by putting $R=0$: and eliminating T between (1), (2), (4), and also between (1), (3), (4), we have the two following equations to the form of the string.

$$\left.\begin{aligned} v\,(d_s x d_s^2 y - d_s y d_s^2 x) &= Y d_s x - X d_s y \\ v\,(d_s x d_s^2 z - d_s z d_s^2 x) &= Z d_s x - X d_s z \end{aligned}\right\}\ldots\ldots\ldots\ldots (B).$$

266. COR. In the case of gravity, supposing the axis of z vertical and measured upwards, we have

$$X=0, \quad Y=0, \quad Z=-g;$$

$$\therefore\ d_s x d_s^2 y - d_s y d_s^2 x = 0.$$

This is the differential equation to a straight line in the plane xy, so that the chain hangs in a vertical plane. Take this plane for the plane of xz, and the lowest point as origin. We have, since

$$d_s T = g d_s z, \text{ and } \therefore\ T = g\,(z+c),$$

$$(z+c)\,(d_s x d_s^2 z - d_s z d_s^2 x) = d_s x,$$

$$\text{or since } d_s x d_s^2 x + d_s z d_s^2 z = 0,$$

$$\frac{d_s z}{z+c} + \frac{d_s^2 x}{d_s x} = 0;$$

$$\text{or } z + c = c d_x s; \text{ since when } z=0,\ d_s x = 1.$$

Hence $d_z x = \dfrac{c}{\sqrt{(z+c)^2 - c^2}}$;

$$\therefore\ x + C = c \log_e \{z + c + \sqrt{(z+c)^2 - c^2}\},$$

and when $z=0$, $x=0$; $\therefore\ C = c\log_e c$; and

$$\therefore\ \epsilon^{\frac{x}{c}} = \frac{z+c}{c} + \sqrt{\left(\frac{z+c}{c}\right)^2 - 1};$$

$$\therefore\ \epsilon^{-\frac{x}{c}} = \frac{z+c}{c} - \sqrt{\left(\frac{z+c}{c}\right)^2 - 1};$$

$$\therefore\ z + c = \frac{c}{2}\left(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right).$$

267. Cor. 2. *The Catenary also possesses the property of being the curve of total minimum tension, supposing gravity alone to act.*

$$\text{Thus tension} = g\,(z + c).$$

Hence to find the curve having the above property, we have

$$\textstyle\int_x (z + c)\, d_x s = \text{a minimum},$$

$$\text{or when } \textstyle\int_x (z + c)\,.\,\sqrt{1 + (d_x z)^2} = \text{a minimum}.$$

This is the case when

$$(z + c)\sqrt{1 + (d_x z)^2} = \frac{(z + c)\,(d_x z)^2}{\sqrt{1 + (d_x z)^2}} + (C = c),$$

by the principles of the Calculus of Variations; or when $z + c = c d_x s$, and this has been just shewn to be the differential equation to the Catenary.

CHAPTER XI.

PROBLEMS.

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1. *Given the magnitudes of two forces which act on a point, and the angle between the lines in which they act; to find the magnitude of their resultant.*

Let P, Q be the two forces acting upon the point O (fig. 81) in the directions OA, OB. Take OA, OB to represent them, and complete the parallelogram $OBCA$; the diagonal represents their resultant R.

Let $\alpha = AOB$ the given angle. Then from the triangle OAC we have

$$OC^2 = OA^2 - 2\,OA\,.\,AC\cos OAC + AC^2$$

$$= OA^2 + 2\,OA\,.\,OB\,.\cos POQ + OB^2;$$

$$\therefore\; R^2 = P^2 + 2PQ\cos\alpha + Q^2.$$

2. *Three forces acting on a point are found to balance each other when their directions make angles* 105^0, 120^0, 135^0 *with each other. Find the relation of the forces to each other.*

Let F_1, F_2, F_3 be the forces respectively opposite to the angles 105^0, 120^0, 135^0. Then by Art. 28, we have

$$F_1 : F_2 : F_3 :: \sin 105^0 : \sin 120^0 : \sin 135^0$$

$$:: \cos 15^0 : \cos 30^0 : \cos 45^0$$

$$:: \cos(45^0 - 30^0) : \cos 30^0 : \cos 45^0$$

$$:: \frac{\sqrt{3}+1}{2\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{1}{\sqrt{2}};$$

$$\therefore \frac{F_1}{\sqrt{3}+1} = \frac{F_2}{\sqrt{6}} = \frac{F_3}{2},$$

for which when the magnitude of one of the forces is given, the magnitudes of the other two are known.

3. *A weight* W *is sustained upon a smooth inclined plane by three forces each equal to* $\frac{1}{3}$W; *one acting vertically upwards, another parallel to the plane, and the third in a horizontal direction; required the inclination of the plane to the horizon.* (Fig. 82.)

Let C be the body placed on the inclined plane AB; F_1, F_2, F_3 the forces mentioned in the question. Besides these, C is acted on by gravity which is equal to W and acts in the downwards direction CW, and by the re-action R of the plane, which because the plane is smooth, acts in a direction CR perpendicular to AB. Hence the body C is kept at rest by five forces, all of which act in the same plane; hence the conditions of equilibrium are (Art. 32), that the sums of the resolved parts of these forces parallel to two lines in the plane of the forces shall be separately equal to zero. Resolve them in the directions of CB, CR;

$$\therefore\ 0 = R\cos RCB + F_1 \cos F_1CB + F_2 \cos F_2CB + F_3 \cos F_3CB + W\cos WCB,$$

and

$$0 = R\cos RCR + F_1 \cos F_1CR + F_2 \cos F_2CR + F_3 \cos F_3CR + W\cos WCR.$$

If we put θ for BCF_3 the required inclination of the plane, these equations become

$$0 = F_1 \sin\theta + F_2 + F_3 \cos\theta - W\sin\theta,$$

$$\text{and } 0 = R + F_1\cos\theta - F_3\sin\theta - W\cos\theta \;\ldots\ldots\ldots\; (A).$$

The former of these, observing that $F_1 = F_2 = F_3 = \frac{1}{3}W$, gives

$$1 + \cos\theta = 2\sin\theta,$$

$$\text{or } 2\cos^2\frac{\theta}{2} = 4\sin\frac{\theta}{2}\cos\frac{\theta}{2};$$

$$\therefore \tan\frac{\theta}{2} = \tfrac{1}{2};$$

$$\therefore \theta = 2\tan^{-1}(\tfrac{1}{2})$$
$$= 53^\circ . 7' . 48''.$$

REMARK. The reader will observe that we have obtained this result without making use of the second condition (A) of equilibrium. From that equation we might have determined R, the pressure of the body upon the plane; but as that is not required by the enunciation of the problem, we make the important remark, that it is not always necessary to employ *all* the equations of equilibrium in solving a problem: and in resolving forces, our aim must be to resolve them in a direction at right angles to such forces as are not known and not required to be found.

The directions in which we may resolve the forces are quite arbitrary (Art. 98); we might, for instance, have resolved the forces parallel and perpendicular to the horizon, from which would have resulted the two equations

$$0 = R\sin\theta - F_2\cos\theta - F_3,$$
$$\text{and } 0 = R\cos\theta + F_1 + F_2\sin\theta - W;$$

but here we see the unknown force R is involved in both the equations of equilibrium; and in order to solve the proposed problem it is necessary to eliminate R between them: this necessity is avoided, and the result at once obtained, by resolving the forces in a direction at right angles to that in which R acts. We shall generally avail ourselves of this artifice in the problems which follow.

4. *Three equal forces act upon a point in the directions of three lines which include angles* 105°, 120°, 135°; *find the magnitude and position of their resultant.*

Since the sum of the given angles = 360°, the forces all act in one plane. Let $F_1 F_2 F_3$ (each equal to P) be the three forces acting on the point O (fig. 83), the angles F_1OF_2, F_1OF_3 being 120^0, 135^0 respectively. Produce F_1O to x, and in the plane of the forces draw Oy perpendicular to Ox. Then if R be the resultant, and θ the angle which its direction makes with Ox, we have, proceeding as in Art. 31,

$$\begin{aligned} R\cos\theta &= -F_1 + F_2\cos F_2Ox + F_3\cos F_3Ox \\ &= -P + P\cos 60^0 + P\cos 45^0 \\ &= P\,.\,\frac{\sqrt{2}-1}{2}, \end{aligned}$$

$$\begin{aligned} R\sin\theta &= F_2\sin 60^0 - F_3\sin 45^0 \\ &= P\,.\,\frac{\sqrt{3}-\sqrt{2}}{2}; \end{aligned}$$

$$\therefore\ \tan\theta = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1} = \cdot 7673269;$$

$$\therefore\ \theta = 37^0\,.\,30',$$

which determines the position of R; and the equation

$$\begin{aligned} R^2 &= (R\cos\theta)^2 + (R\sin\theta)^2 \\ &= P^2\left(2 - \frac{\sqrt{6}+\sqrt{2}}{2}\right) \\ &= P^2 \times \cdot 0681484, \end{aligned}$$

$$\text{or}\ \ R = P \times \cdot 2610525,$$

gives the magnitude of R.

5. *If three forces proportional to the sides of a triangle be applied perpendicularly at their middle points, they will balance, supposing them all to act in the plane of the triangle.*

Let ABC (fig. 84) be the triangle; a, b, c the middle points of its sides. At these points apply three forces F_1, F_2, F_3 respectively proportional to the sides on which they act, in directions perpendicular to those sides. Then, because the sides of the triangle are bisected perpendicularly, the lines F_1a, F_2b, F_3c being produced will meet in O the centre of the circle circumscribing the triangle. We may therefore suppose them to act at O: and because

$$F_1 : F_2 : F_3 :: BC : AC : AB$$
$$:: \sin A : \sin B : \sin C$$
$$:: \sin F_2OF_3 : \sin F_1OF_3 : \sin F_1OF_3, .$$

therefore, by the converse of Art. 28, the forces balance each other.

6. *Two given equal uniform beams* AC, BC (fig. 85) *having their lower ends connected by a string are placed in a vertical plane, upon a smooth floor, their upper ends leaning against each other. Required the tension of the string* AB.

Let a be the length of each beam, b the length of the string AB; $\theta = \angle CAB$; G the centre of gravity of AC, which since the beam is uniform will be in its middle point. The beam AC is kept at rest by R the pressure of BC against it in a horizontal direction; T the tension of the string AB in the direction AB; R' the reaction of the floor, which since the floor is smooth will act at right angles to AB; and by W the weight of the beam acting at G in the direction GW vertically downwards. (The conditions of equilibrium for this case are stated in Art. 67.)

To avoid the force R', resolve the forces horizontally, and take the moments about A; then

$$0 = R - T,$$

$$\text{and } 0 = R \,.\, AC \sin\theta - W \,.\, AG \cos\theta;$$

$$\therefore\ T = R = W\frac{AG}{AC} \,.\, \cot\theta$$

$$= \frac{W}{2} \,.\, \frac{b}{\sqrt{4a^2 - b^2}}.$$

7. *A string* PCP (fig. 86) *having two equal weights* P, P *fastened to its extremities, passes over a pulley* C, *and two pegs* A, B. *A smooth heavy ring* Q *is passed over* C: *required the position in which it will rest, its inner diameter being such as to*

keep the parts of the string above it parallel; and the pegs A, B *being similarly situated with respect to* C.

Let W represent the weight of the ring; θ the inclination of AQ to the horizon. Because the ring is smooth the tension of the string will be equal to P in every part; and when the equilibrium is established we may suppose the ring and string to cohere at the points of contact, by which supposition we perceive that Q is pulled upwards by the two parts of the string between the ring and the pulley; and obliquely by the portions of the string between the ring and the pegs. Hence the ring is kept at rest by five forces, viz. the two vertical forces P, P acting along the lines QC, $Q'C$; the two oblique forces P, P acting along the lines QA, $Q'B$; and its own weight W acting vertically downwards: consequently resolving them in a vertical direction, we have

$$0 = 2P - 2P \sin\theta - W;$$

$$\therefore \sin\theta = 1 - \frac{W}{2P};$$

which determines the position of Q.

8. *Two weights* P, Q (fig. 87) *are connected by a string, which passes over two smooth pegs* A, B *situated in a horizontal line, and supports a weight* W *which hangs from a smooth ring, through which the string passes. Find the position of equilibrium: and also whether the equilibrium is stable or unstable.*

Since the ring C is smooth, the tension of the string is the same throughout, and therefore

$$P = Q.$$

Also we may consider the point C as kept at rest by three forces; the tensions of CA, CB, and the weight W; hence by Art. 28,

$$\frac{P}{\sin BCW} = \frac{P}{\sin ACW} = \frac{W}{\sin ACB};$$

$$\therefore ACW = BCW;$$

$$\text{and } \frac{W}{P} = \frac{\sin ACB}{\sin ACW}$$

$$= \frac{\sin \theta}{\sin \frac{\theta}{2}}$$

$$= 2 \cos \frac{\theta}{2},$$

θ representing the angle ACB. This equation gives θ, from which the place of the ring is known.

Again, let $2a = AB$, $2b =$ the length of the string, $c = CW$, $\bar{z}$ the distance of the common centre of gravity of P, Q, W below AB; then producing WC to meet AB in D, we have

$$(P + Q + W)\,\bar{z} = P \cdot AP + Q \cdot BQ + W \cdot DW;$$

$$\therefore\ (2P + W)\,\bar{z} = 2P \cdot \left(b - \frac{a}{\sin \frac{\theta}{2}}\right) + W \cdot \left(c + a \cot \frac{\theta}{2}\right);$$

$$\therefore\ (2P + W)\, d_\theta \bar{z} = Pa \operatorname{cosec} \frac{\theta}{2} \cot \frac{\theta}{2} - \frac{1}{2} Wa \operatorname{cosec}^2 \frac{\theta}{2}$$

$$= \frac{a}{2} \operatorname{cosec}^2 \frac{\theta}{2} \left(2P \cos \frac{\theta}{2} - W\right) = 0; \text{ (Art. 168)};$$

$$\therefore\ (2P + W)\, d_\theta^2 \bar{z} = -\frac{a}{2} \operatorname{cosec}^2 \frac{\theta}{2} \cdot P \sin \frac{\theta}{2}, \text{ when } W = 2P \cos \frac{\theta}{2}.$$

Hence, in the position of equilibrium $\bar{z}$ is a maximum, and therefore the altitude of the common centre of gravity is a minimum, consequently (Art. 169) the position is one of stable equilibrium.

Remark. The equation $d_\theta \bar{z} = 0$, gives $W = 2P \cos \frac{\theta}{2}$ for the condition of equilibrium, and consequently the latter part of the preceding investigation includes the solution of the whole question proposed; the first part therefore might have been omitted; but we have inserted it as an example of the application of Art. 28.

9. *A straight rod rests with its ends upon two given smooth inclined planes, in such a position that the vertical plane which passes through the rod is at right angles to the given planes; find the position of equilibrium of the rod.*

Let AB (fig. 88) be the rod, G its centre of gravity; α, β, θ the inclinations of the planes OA, OB, and of the rod to the horizon. R, R' the reactions of the planes at A, B, which will be in normal directions because the planes are smooth. $AG = a$, $BG = b$. The forces which keep the rod in equilibrium are the reactions R, R' and its own weight. To avoid the weight of the rod, resolve the forces parallel to the horizon; and take their moments about G;

$$\therefore\ 0 = R\sin\alpha - R'\sin\beta,$$

$$\begin{aligned} \text{and} \quad 0 &= R\,.\,GA\sin GAR - R'\,.\,GB\sin GBR' \\ &= Ra\cos OAB - R'b\cos OBA \\ &= Ra\cos(\alpha+\theta) - R'b\cos(\beta-\theta); \end{aligned}$$

$$\therefore\ \frac{a\cos(\alpha+\theta)}{\sin\alpha} = \frac{b\cos(\beta-\theta)}{\sin\beta};$$

$$\text{and } \therefore\ \tan\theta = \frac{a\cot\alpha - b\cot\beta}{a+b}.$$

Remark. In the enunciation of the preceding problem it is assumed, that the vertical plane which passes through the rod is at right angles to the given planes. This is a particular case of a more general proposition which we shall now give.

10. *If a body of any form whatever rest in equilibrium upon two smooth inclined planes, the line of intersection of the planes must be horizontal.*

For distinctness' sake let the planes be called A and B. Then as they are smooth the reactions of the plane A, at the various points of contact of the body with it, are all perpendicular to the plane and therefore parallel to one another; and consequently their resultant (R suppose) is (Art. 71) also perpendicular to A.

Similarly (R') the resultant of the reactions of the plane B upon the body is perpendicular to B. Consequently we may consider the body as kept at rest by the action of three forces; viz. R and R' (acting respectively perpendicular to the planes A and B) and the weight (W) of the body acting vertically at its centre of gravity. Consequently, by Art. 96, the lines in which these forces act lie in one plane; and as the vertical through the centre of gravity is in this plane, therefore the plane is a vertical plane; and as normals to the planes A, B, in which R and R' act also lie in it, therefore it is perpendicular to each of them, and consequently also to their intersection: that is, the line of intersection of A and B is perpendicular to a vertical plane, and consequently it is horizontal.

Cor. If a body of any form rest in equilibrium upon two given inclined planes with one point only in contact with each plane, the vertical plane which passes through the two points of contact will be at right angles to the given planes, and pass through the centre of gravity of the body.

11. LM *is a smooth sphere of radius* r (6 *inches*) *and weight* w ($3\frac{1}{2}$ *lbs.*), *in contact with a plane* AM *inclined to the horizon at an angle* α (60°). AB *is a beam of weight* W (100 *lbs.*), *and length* a (6 *feet*), *moveable about a hinge at* A, *and by its pressure preventing the sphere from descending down the plane. Determine the positions of the beam and sphere.* (Fig. 89.)

Let R be the reaction at L between the sphere and beam; and R' that at M between the sphere and inclined plane; since the sphere is smooth, the former acts in a direction perpendicular to AB; and the latter in a direction perpendicular to AM. Let $2\theta = \angle BAM$.

We may consider AB as a lever, whose fulcrum is A, kept at rest by R at L in the direction CL; and W at G, the centre of gravity of AB, in a vertical direction;

$$\therefore\ R \,.\, AL = W \,.\, \frac{a}{2} \cos(\alpha + 2\theta),$$

$$\text{or}\ \ R \,.\, r \cot\theta = W \,.\, \frac{a}{2} \cos(\alpha + 2\theta).$$

The sphere is kept in equilibrium by its own weight acting downwards at C, and the reactions RR' in the directions LC, MC. To avoid R' resolve these forces parallel to the plane;

$$\therefore\ 0 = R \sin 2\theta - w \sin \alpha.$$

Hence, eliminating R,

$$2wr \sin \alpha \cot \theta = Wa \sin 2\theta \cos (\alpha + 2\theta);$$

$$\therefore\ \sin^2 \theta \cos (\alpha + 2\theta) = \frac{wr}{Wa} \cdot \sin \alpha.$$

By substituting for W, w, a, r, α their values, we find from this equation

$$\theta = 4^\circ . 45' . 30'';$$

$$\text{and } \therefore\ \alpha + 2\theta = 69^\circ . 49',$$

which is the inclination of the beam to the horizon.

The position of the ball is known from the equation

$$AM = r \cot \theta$$
$$= 5{\cdot}822314 \text{ feet}.$$

12. *A uniform heavy rod* CD *rests with one end* D *on a smooth inclined plane* DB, *and the other is suspended by a string of given length from a fixed point* A. *Find the position of equilibrium.* (Fig. 90.)

Draw AB perpendicular to the plane; and let ϕ, θ be the angles which AC, CD respectively make with AB; let α be the inclination of the plane DB to the horizon; let G be the centre of gravity of the rod; $a = CG = DG$, $b = AC$, $c = AB$, R the reaction of the plane at D, which since the plane is smooth will be in a normal direction; $T =$ the tension of the string CA. Since $AB =$ the sum of the projections of AC, CD upon it;

$$\therefore\ c = b \cos \phi + 2a \cos \theta \ \ldots\ldots\ldots\ldots\ (1).$$

To avoid the weight of the rod, resolve the forces in a horizontal direction, and take their moments about G;

$$\therefore\ 0 = R \sin \alpha - T \sin (\phi - \alpha),$$
$$\text{and } 0 = Ra \sin \theta - Ta \sin (\theta - \phi);$$

$$\therefore \ \sin\theta \sin(\phi - \alpha) = \sin\alpha \sin(\theta - \phi);$$

$$\therefore \ 2\cot\phi = \cot\theta + \cot\alpha \ \ldots\ldots\ldots\ldots\ldots\ldots \ (2).$$

But $b\cos\phi = c - 2a\cos\theta$ from (1);

$$\therefore \ 2b\sin\phi = (c - 2a\cos\theta)(\cot\theta + \cot\alpha);$$

$$\therefore \ 4b^2 = (2b\cos\phi)^2 + (2b\sin\phi)^2$$

$$= (c - 2a\cos\theta)^2 \{4 + (\cot\theta + \cot\alpha)^2\}.$$

From this equation θ must be found by approximation, and then ϕ will be known from (2).

13. *Three rods* AD, AE, BC *are connected by hinges at* A, B, C; AE *is vertical and fixed at* E, *and* AD *horizontal. At* D *a given weight* W *is suspended. Find the strain upon the hinges.* (Fig. 91.)

Since the rod BC has hinges at both ends, it is incapable of exerting any action except in direction of its length; let T be the pushing force which it exerts upon the hinge B in the direction CB, and upon C in the direction BC. Let the strain upon the hinge A be resolved into the forces X, Y in the directions BA, AC. Let

$$\alpha = \angle ABC, \ a = BC, \ b = BD.$$

Then the rod AD is kept at rest by X, Y, T and W; resolve them vertically and horizontally, and take their moments about B;

$$\therefore \ 0 = W - T\sin\alpha + Y,$$

$$0 = T\cos\alpha - X,$$

$$\text{and } \ 0 = Wb - Ya\cos\alpha.$$

From these three equations we find,

$$Y = \frac{Wb}{a\cos\alpha},$$

$$T = W \cdot \frac{a\cos\alpha + b}{a\cos\alpha\sin\alpha},$$

$$X = W \cdot \frac{a\cos\alpha + b}{a\sin\alpha}.$$

The first and last of these determine the magnitude and direction of the strain upon the hinge A; and the second equation gives the magnitude of the strain upon B or C; the direction of this strain has been stated already to be CB for B, and BC for C.

If the joints at A, B, C were rigid, the action of BC not being necessarily in the direction of its length would be indeterminate: BC might even be removed without affecting the equilibrium.

14. AB *is a heavy uniform rod, moveable in a vertical plane about a hinge* A; *a given weight* P *sustains the rod by means of a string* BCP *passing over a smooth pin* C *situated in a vertical through* A *and at a distance* AC = AB. *Find the position of equilibrium of the rod by the principle of virtual velocities.* (Fig. 92.)

Let W be the weight of the rod, $a = AB$ its length, Q any point in it; draw QM perpendicular to AC. $x = AQ$, $\theta = \angle BAC$. Then the virtual velocity of P

$$= d\,.\,CP = d\,(BCP - BC) = d\left(BCP - 2a\sin\frac{\theta}{2}\right)$$

$$= -d\,.\,2a\sin\frac{\theta}{2} = -a\cos\frac{\theta}{2}\,.\,d\theta.$$

The weight of a small element of the rod at Q, the length of which is equal to $\delta x = W\dfrac{\delta x}{a}$; and the virtual velocity of this

$$= d\,.\,CM = d\,(a - x\cos\theta) = x\sin\theta d\theta.$$

Hence for the whole rod the value of $\Sigma\,(Fds)$ (Art. 113)

$$= \Sigma\left(W\frac{\delta x}{a}\,.\,x\sin\theta d\theta\right)$$

$$= \frac{W}{a}\sin\theta d\theta\,.\,\Sigma\,(x\delta x)$$

$$= \frac{W}{a}\sin\theta d\theta \int x dx,\ \text{from } x = 0 \text{ to } x = a$$

$$= \tfrac{1}{2}Wa\sin\theta d\theta.$$

Hence, by the principle of virtual velocities,

$$0 = P\left(-a\cos\frac{\theta}{2}\,d\theta\right) + \tfrac{1}{2}Wa\sin\theta d\theta;$$

$$\therefore\ \sin\frac{\theta}{2} = \frac{P}{W}.$$

REMARK. The preceding solution is more difficult than is absolutely necessary; we preferred giving it however as an illustration of the meaning of the symbol Σ in Art. 113. The following is more simple.

We may consider the weight of the beam as being collected at its centre of gravity. Let Q be this point. Then by the principle of virtual velocities,

$$\begin{aligned} 0 &= P.d.CP + W.d.CM \\ &= P\left(-a\cos\frac{\theta}{2}\,d\theta\right) + W.d\left(a - \frac{a}{2}\cos\theta\right) \\ &= -Pa\cos\frac{\theta}{2}\,d\theta + W\frac{a}{2}\sin\theta d\theta, \end{aligned}$$

the same result as before.

15. *Two heavy particles* P, Q *are connected by an inflexible rod; and one of them* (P) *rests upon a given smooth inclined plane. Required the nature of the curve on which the other* (Q) *must rest, that there may be equilibrium in all positions.*

Since there is equilibrium in all positions the common centre of gravity of the bodies neither ascends nor descends (Art. 169), it is therefore always at the same height above a given horizontal plane. Let the equation of the given inclined plane be

$$y' = mx' \quad\quad (1),$$

the axis of x being horizontal, and that of y vertical. Let $x'y'$ be the co-ordinates of P, and xy those of Q. Then denoting the altitude of the common centre of gravity of P and Q above the axis of x by b, and the length of the rod by a, we have

$$(P + Q)\,b = Py' + Qy \quad\quad (2),$$

$$a^2 = (x - x')^2 + (y - y')^2 \quad\quad (3).$$

From (1) and (2) we find

$$y' = \left(1 + \frac{Q}{P}\right) b - \frac{Q}{P} y,$$

$$\text{and} \quad x' = \left(1 + \frac{Q}{P}\right) \frac{b}{m} - \frac{Qy}{Pm},$$

which being substituted in (3), give the following equation of the required curve:

$$0 = x^2 + \frac{2Q}{mP} xy + \left\{\left(1 + \frac{Q}{P}\right)^2 + \frac{Q^2}{m^2P^2}\right\} y^2 - \frac{2}{m}\left(1 + \frac{Q}{P}\right) bx$$

$$- 2\left(1 + \frac{Q}{P}\right)\left\{1 + \left(1 + \frac{1}{m^2}\right)\frac{Q}{P}\right\} by + \left(1 + \frac{1}{m^2}\right)\left(1 + \frac{Q}{P}\right)^2 - a^2.$$

The values of the coefficients of the first three terms shew that the required curve is an ellipse.

16. *A rod* AB *is placed in a smooth hemispherical bowl, so as to lean against the edge of the bowl at* P, *with one end* A *within it. Find the position of equilibrium of the rod.* (Fig. 93.)

Let C be the centre of the bowl, G the centre of gravity of the rod. The rod is kept in equilibrium by the reaction of the bowl at A, the direction of which passes through C; by the reaction of the edge of the bowl at P, which will be in a line PQ at right angles to AB; and by its own weight acting vertically at G. There being but three forces, their directions all pass through a point (Art. 96); hence QG is vertical, and

$$\frac{AQ}{AG} = \frac{\sin AGQ}{\sin AQG}.$$

Let $AG = a$, and $r =$ the radius of the bowl; then because APQ is a right angle, AQ is a diameter of the sphere, and therefore $= 2r$; also if $\theta = CPA$, the inclination of the rod to the horizon,

$$AGQ = \pi - PGQ = \pi - \left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2} + \theta,$$

$$\text{and } AQG = PGQ - PAC = \left(\frac{\pi}{2} - \theta\right) - \theta = \frac{\pi}{2} - 2\theta;$$

$$\therefore \frac{2r}{a} = \frac{\sin\left(\frac{\pi}{2} + \theta\right)}{\sin\left(\frac{\pi}{2} - 2\theta\right)} = \frac{\cos\theta}{\cos 2\theta}$$

$$= \frac{\cos\theta}{2\cos^2\theta - 1};$$

$$\therefore \cos^2\theta - \frac{a}{4r}\cos\theta = \tfrac{1}{2};$$

from which equation θ is known.

17. *A smooth heavy rod* AB *moveable in a vertical plane about a hinge at* A, *leans against a heavy prop* CD *also moveable in the same plane about a hinge at* C. *Find the position of equilibrium.* (Fig. 94.)

Let G, G' be the centres of gravity of the two rods, at which points we may suppose their weights W, W' to be applied. Let R' be the pressure which the rod AB exerts against the prop; R the reaction of the prop against AB; these forces will be equal and opposite, and act in a line perpendicular to AB.

The rod AB is a lever whose fulcrum is A, kept in equilibrium by R and W; hence putting $AG = a$, $CD = b$, $AC = c$, and the angles BAC, $ACD = \theta$, ϕ respectively, we have by taking the moments of R and W about A,

$$W \cdot a\cos\theta = R \cdot AD$$

$$= Rb \cdot \frac{\sin\phi}{\sin\theta} \quad \text{.................} \quad (1).$$

Similarly we perceive that the prop CD is a lever whose fulcrum is C, kept in equilibrium by R' and W', hence taking the moments of these forces about C, we have

$$W' \cdot \frac{b}{2} \cos\phi = R'b \sin CDR'$$

$$= Rb \cos CDA$$

$$= - Rb \cos(\theta + \phi).$$

Hence eliminating R by means of (1), we find

$$0 = 2Wa \cos\theta \sin\theta \cos(\theta + \phi) + W'b \cos\phi \sin\phi.$$

Also from the geometrical property of the figure,

$$\frac{c}{b} = \frac{\sin(\theta + \phi)}{\sin\theta}.$$

The last two equations will give the values of θ and ϕ.

18. *Two rods* AB, AC *rest against each other upon the horizontal plane* ED *at* A, *and against two smooth vertical parallel walls at* B, C; *given the lengths and weights of the rods, to find the distance of the walls when the angle between the rods is a right angle.* (Fig. 95.)

Let a, b be the lengths, and W, W' be the weights of the rods AB, AC, which we may suppose acting at their centres of gravity. Let R, R' perpendicular to EB, DC be the reactions of the vertical walls. $\theta = BAE$. Then the rod AB is kept in equilibrium by W, R, and the forces which act upon it at A. To avoid these last take the moments of all the forces which act on AB about A;

$$\therefore\ 0 = W \cdot \frac{a}{2} \cos\theta - Ra \sin\theta.$$

Similarly for the beam AC,

$$0 = W' \cdot \frac{b}{2} \sin\theta - R'b \cos\theta.$$

To obtain an equation between R and R', not involving the forces at A, let us suppose the rods to become rigidly joined at A, which will not disturb the equilibrium, nor affect R and R'; BAC being now a rigid body kept at rest by R, R' acting horizontally, and its weight and the reaction of ED at A acting in a vertical direction, we have, taking the horizontal forces,

$$0 = R - R'.$$

Hence, eliminating R and R' between the three equations now found, we obtain

$$W \cos^2 \theta = W' \sin^2 \theta;$$

$$\therefore \tan \theta = \left(\frac{W}{W'}\right)^{\frac{1}{2}};$$

$$\text{and } \therefore ED = a \cos \theta + b \sin \theta$$

$$= \frac{a + b \tan \theta}{\sqrt{1 + \tan^2 \theta}}$$

$$= \frac{a \sqrt{W'} + b \sqrt{W}}{\sqrt{W' + W}}.$$

19. *Two given rods connected by a hinge are laid across a smooth horizontal cylinder of given radius. Determine the position of equilibrium and the strain upon the hinge.*

Let AC, BC (fig. 96) be the rods, resting upon the circle whose centre is O at the points P, Q. Let G, G' be their centres of gravity. Join OC, draw OH vertical, and upon it drop the perpendiculars GM, $G'M'$. Let

$$COH = \theta, \; ACO = BCO = \phi, \; GC = a, \quad G'C = b,$$

the radius of the cylinder $= r$, W, W' the weights of the rods. Then if $\bar{z}$ be the altitude of the common centre of gravity of the rods above a horizontal plane passing through the point O,

$$(W + W')\,\bar{z} = W \,.\, OM + W' \,.\, OM'.$$

Now if OH cut AC in H, and BC produced in H',

$$OM = OC \cos \theta - CG \,.\, \cos CHH'$$

$$= \frac{r \cos \theta}{\sin \phi} - a \cos (\phi + \theta),$$

$$\text{and } OM' = OC\cos\theta - CG'\cos H'$$

$$= \frac{r\cos\theta}{\sin\phi} - b\cos(\phi-\theta);$$

$$\therefore\ (W+W')\bar{z} = (W+W')r.\frac{\cos\theta}{\sin\phi}$$

$$-Wa\cos(\phi+\theta) - W'b\cos(\phi-\theta).$$

Now in a position of equilibrium $\bar{z}$, which is a function of the two independent variables θ and ϕ, must be a maximum or a minimum;

$$\therefore\ 0 = (W+W')\,d_\theta\bar{z} = -(W+W')r.\frac{\sin\theta}{\sin\phi}$$

$$+Wa\sin(\phi+\theta) - W'b\sin(\phi-\theta),$$

$$\text{and } 0 = (W+W')\,d_\phi\bar{z} = -(W+W')r.\frac{\cos\theta\cos\phi}{\sin^2\phi}$$

$$+Wa\sin(\phi+\theta) + W'b\sin(\phi-\theta);$$

$$\therefore\ (W+W')r\operatorname{cosec}^2\phi = (Wa-W'b)\cot\theta + (Wa+W'b)\cot\phi,$$

$$\text{and } (W+W')r\operatorname{cosec}^2\phi = (Wa-W'b)\tan\theta + (Wa+W'b)\tan\phi.$$

From which we find, by subtraction,

$$\frac{\tan 2\theta}{\tan 2\phi} = \frac{W'b - Wa}{W'b + Wa} \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$

and by eliminating θ,

$$4WW'ab\sin^4\phi - (W+W')(Wa+W'b)r\tan\phi + (W+W')^2r^2 = 0;$$

from which ϕ being found by approximation, θ will be known from (1).

To find the strain upon the hinge.

Let T be the strain, exerted upon AC in the direction CT, and upon CB in the opposite direction. To avoid the reaction at P, which is unknown, resolve the forces, which keep AC at rest, parallel to AC, and take their moments about P;

$$\therefore\ 0 = T\cos(\pi - ACT) - W\cos AHO,$$

$$\text{and } 0 = T.PC\sin ACT - W.PG.\sin AHO.$$

From these equations we find

$$\left(\frac{T}{W}\right)^2 = \cos^2 AHO + \frac{PG^2}{PC^2} . \sin^2 AHO$$

$$= \cos^2(\phi + \theta) + \left(\frac{a}{r}\tan\phi - 1\right)^2 \sin^2(\phi + \theta),$$

from which T is known; and

$$\tan ACT = -\frac{PG}{PC} . \tan AHO$$

$$= \left(1 - \frac{a}{r} . \tan\phi\right) \tan(\phi + \theta),$$

gives the direction in which T acts.

20. *A given weight* W *is sustained on a given inclined plane, partly by friction and partly by a power* P *acting in a given direction. Find the greatest and least values of* P.

Let C (fig. 97) be the body placed on the inclined plane AB; let R be the reaction of the plane in a normal direction, and μ the coefficient of friction between the body and the plane: then if the tendency of C is to slide *down* the plane, P having its *least* value, the friction μR will act in the direction CB to prevent the motion; and therefore resolving the forces parallel and perpendicular to AB,

$$0 = \mu R + P\cos\theta - W\sin i,$$

$$\text{and } 0 = R + P\sin\theta - W\cos i,$$

i representing the inclination of the plane to the horizon, and θ the $\angle PCB$. Hence eliminating R,

$$P = W . \frac{\sin i - \mu\cos i}{\cos\theta - \mu\sin\theta}.$$

This is the *least* value of P; i.e. if P be less than this, C will begin to slide *down* the plane.

If P have its greatest value, C will be on the point of moving *up* the plane, and therefore the friction μR will act *down* the plane; this will be taken account of by writing $-\mu$ for μ in the preceding result; consequently the greatest limit of P

$$=W\cdot\frac{\sin i+\mu\cos i}{\cos\theta+\mu\sin\theta}.$$

Any value of P between these two limits will maintain equilibrium.

COR. The limiting values of P found above may be put under the forms

$$W\cdot\frac{\sin(i-\tan^{-1}\mu)}{\cos(\theta+\tan^{-1}\mu)};\quad \text{and } W\cdot\frac{\sin(i+\tan^{-1}\mu)}{\cos(\theta-\tan^{-1}\mu)};$$

and, from comparing which with Art. 212, we perceive that the least and greatest values of P are such as would balance W if the inclined plane were smooth and its inclination *diminished* or *increased* respectively by the angle $\tan^{-1}\mu$.

21. *To find the limiting values of* P *in the common screw when friction acts.*

Let W be the weight sustained, $i=$ the inclination of the thread of the screw to the horizon; $R=$ the reaction perpendicular to the thread, $\mu R=$ the friction along the thread: and suppose that P has its *least* value. Let $r=$ the radius of the screw, and p the length of the arm by which P acts; then resolving all the forces vertically, and taking their moments about the axis of the screw (μR acts *up* the thread), since the axis of the screw is only moveable lengthwise, by Art. 94 we have for equilibrium

$$0=R\cos i+\mu R\sin i-W,$$

$$0=(R\sin i-\mu R\cos i)\,r-Pp.$$

By eliminating R between these equations, we find

$$P=W\cdot\frac{r}{p}\cdot\frac{\tan i-\mu}{1+\mu\tan i}$$

$$=W\cdot\frac{r}{p}\cdot\tan(i-\tan^{-1}\mu).$$

Hence the *least* value of P is the same as in a screw without friction, the thread of which is inclined to the horizon at the angle

$$i - \tan^{-1}\mu.$$

By writing $-\mu$ for μ, we find that the greatest value of P is the same as in a screw without friction, the thread of which is inclined to the horizon at the angle

$$i + \tan^{-1}\mu.$$

22. *Let* AC *be a curve line in a vertical plane;* P, Q *given weights attached to the extremities of a string which passes over a smooth pin at* B*; to shew how to find the position of equilibrium.* (Fig. 98.)

Take the vertical line Bx for the axis of x; and any fixed point A in it for the origin of co-ordinates: draw QM perpendicular to Bx; and put $AM = x$, $QM = y$, $Bp = x'$; then if P descend through a small space dx', the corresponding space descended by Q will be dx; and as P and Q are acted on by no other forces than gravity, except the tension of the string and reaction of the curve line, the virtual velocities of P and Q are dx' and dx; and consequently, by Art. 113,

$$Pdx' + Qdx = 0;$$

this is the only mechanical condition of equilibrium. The geometrical nature of the machine is expressed by the equation of the curve

$$y = F(x),$$

and the equation, (b being the length of the string, and a denoting AB)

$$b = x' + \sqrt{(a+x)^2 + y^2}.$$

Ex. *Let* AC *be a parabola, and* B *the point where the axis intersects the directrix.*

In this case $y^2 = 4ax$;

$$\therefore\ b = x' + \sqrt{(a+x)^2 + 4ax};$$

$$\therefore\ 0 = dx' + \frac{3a + x}{(a^2 + 6ax + x^2)^{\frac{1}{2}}} \cdot dx$$

$$= -\frac{Q}{P} \cdot dx + \frac{3a + x}{(a^2 + 6ax + x^2)^{\frac{1}{2}}} \cdot dx.$$

Hence dividing by dx, and reducing the result, we find

$$x = -3a + \frac{2a\sqrt{2}}{\left(1 - \frac{P^2}{Q^2}\right)^{\frac{1}{2}}}.$$

23. *The weight* P *in* Prob. 21 *instead of hanging perpendicularly, rests upon a given curve line* AD*; to find the position of equilibrium.* (Fig. 99.)

If x', y' be the co-ordinates of P, and x, y those of Q, both measured from B as origin, the virtual velocities of P and Q will be respectively dx' and dx; consequently

$$Pdx' + Qdx = 0.$$

To this we must join the equations of the two curves

$$y' = F'(x'), \qquad \text{and } y = F(x),$$

and the equation

$$b = \sqrt{x'^2 + y'^2} + \sqrt{x^2 + y^2}.$$

Ex. *Let* AD *be a circle whose centre is in* BA *produced; and* AC *a parabola, the directrix of which passes through* B.

Then the equation of AC is

$$y^2 = 4a(x - a),$$

and that of AD is

$$y'^2 = 2c(x' - a) - (x' - a)^2,$$

c being the radius of the circle;

$$\therefore\ b = \sqrt{x'^2 + y'^2} + \sqrt{x^2 + y^2}$$

$$= \sqrt{2x'(c+a) - 2ac - a^2} + \sqrt{x^2 + 4ax - 4a^2};$$

$$\therefore\ 0 = \frac{(c+a)\,dx'}{\sqrt{2x'(c+a) - 2ac - a^2}} + \frac{(x+2a)\,dx}{\sqrt{x^2 + 4ax - 4a^2}}$$

$$= \frac{(c+a)\,dx'}{BP} + \frac{(x+2a)\,dx}{BQ};$$

$$\therefore\ \frac{Q}{P} = -\frac{dx'}{dx} = \frac{x+2a}{c+a}\,\frac{BP}{BQ}$$

$$= \frac{x+2a}{c+a}\cdot\left(\frac{b}{BQ} - 1\right),$$

from which equation, BQ being known in terms of x, x may be found.

24. *Two given weights* P, Q *are connected by a string* PAQ *which is laid across a horizontal cylinder; to find the position and nature of the equilibrium.* (Fig. 100.)

It is evident the string will lie in a vertical plane perpendicular to the axis of the cylinder. Let C be the centre of the circular section of the cylinder by this plane. Draw CA vertical, and BCD, PM, QN horizontal: join CP, CQ. Then since the length of the string and the radius of the cylinder are given, the angle PCQ is known; let it be denoted by 2α: and let $\alpha+\theta$, $\alpha-\theta$ represent the angles QCA, PCA; and $a = CA$. Then if $\bar{z}$ be the altitude of the common centre of gravity of P and Q above BD, we have

$$(P+Q)\,z = P\,.\,CM + Q\,.\,CN$$
$$= Pa\cos(\alpha-\theta) + Qa\cos(\alpha+\theta);$$
$$\therefore\ (P+Q)\,d_\theta\,\bar{z} = Pa\sin(\alpha-\theta) - Qa\sin(\alpha+\theta),$$
$$\text{and}\ (P+Q)\,d_\theta^2\bar{z} = -Pa\cos(\alpha-\theta) - Qa\cos(\alpha+\theta)$$
$$= -(P+Q)\,\bar{z} \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now in the position of equilibrium $d_\theta\bar{z} = 0$, and therefore

$$Pa\sin(\alpha-\theta) = Qa\sin(\alpha+\theta),$$

from which we find

$$\tan\theta = \frac{P-Q}{P+Q}\,.\,\tan\alpha,$$

which gives the position of equilibrium, which is unstable, because equation (1) shews that $\bar{z}$ is then a maximum.

25. *A hollow paraboloid is placed with its vertex downwards and axis vertical; a given rod rests within it, leaning against a pin at the focus, and having its lower end upon the parabolic surface. Find the position of equilibrium.* (Fig. 101.)

Let PQ be the rod, G its centre of gravity, S the focus of the paraboloid, AS its axis, BAC a section of it by a vertical plane passing through the rod; $a = AS$, $b = PG$, $r = PS$, $\theta = ASP$; through S draw LS horizontal, and draw MG vertical; let $\bar{z} = MG$. Then by the nature of the parabola

$$r = \frac{2a}{1 + \cos\theta}, \text{ whence } \cos\theta = \frac{2a}{r} - 1.$$

Also
$$\begin{aligned}\bar{z} = GM &= SG \cos\theta \\ &= (r - b)\cos\theta \\ &= (r - b)\left(\frac{2a}{r} - 1\right) \\ &= 2a - r - \frac{2ab}{r} + b;\end{aligned}$$

$$\therefore\ d_r\bar{z} = \frac{2ab}{r^2} - 1,$$

$$\text{and } d_r^2\bar{z} = -\frac{4ab}{r^3} \qquad (1).$$

In the position of equilibrium $d_r\bar{z} = 0$, and therefore

$$r = \sqrt{2ab};$$

from which the position of the rod is known. Equation (1) shews that the altitude of G is then a minimum; and therefore the position is one of stable equilibrium.

26. *A paraboloid, formed by the revolution of a given parabolic area about its axis, is placed with its convex surface upon a horizontal plane; to find the position in which it will rest.* (Fig. 102.)

Let AC be the axis of the parabola, inclined at an $\angle\,\theta$ to the horizontal plane: P the point on which it rests; draw PN vertical: then since there is equilibrium the centre of gravity must be in the line PN (Art. 132), but it is also in AC, the axis of the parabola, consequently it is at N. Draw PM perpendicular to AC; let $a = AC$, $b = BC$; then the latus rectum $= \dfrac{b^2}{a}$,

$$\text{and } \therefore\ NM = \frac{b^2}{2a};$$

$$\therefore\ MP = \frac{b^2}{2a} \cdot \cot\theta;$$

$$\therefore\ AM = \frac{MP^2}{\left(\frac{b^2}{a}\right)} = \frac{b^2}{4a} \cdot \cot^2\theta;$$

$$\therefore\ AN = AM + MN$$

$$= \frac{b^2}{4a}(\cot^2\theta + 2).$$

But because N is the centre of gravity, $AN = \frac{2}{3}a$. (Ex. 5, Art. 177);

$$\therefore\ \frac{2}{3}a = \frac{b^2}{4a}(\cot^2\theta + 2);$$

$$\therefore\ \cot^2\theta = \frac{8a^2}{3b^2} - 2,$$

from which the position is known.

Cor. The least value of $\cot\theta$ is when $\theta = \frac{\pi}{2}$: hence when

$$\frac{8a^2}{3b^2} \text{ is} = \text{or} < 2,$$

$$\text{or, when } a \text{ is} = \text{or} < \frac{b\sqrt{3}}{2},$$

the solid can only rest in equilibrium with its axis vertical.

27. *Two heavy rods* AC, CB *connected by a hinge at* C *rest on two smooth points* D, E, *situated in a horizontal line: find the position of equilibrium.* (Fig. 103.)

Let G, g be the centres of gravity; and W, W' the weights of the rods AC, BC; R, R' the reactions of the points D, E which will be at right angles to the rods, because the points on which they rest are smooth. Join DE, and let θ, ϕ denote the angles CDE, CED; and put $CG = a$, $Cg = a'$, $DE = b$. The rod AC is kept in equilibrium by three forces, viz. its own weight at G, the reaction R at D, and the tension of the hinge C; to avoid the last, (the magnitude and direction of which are unknown, and are not required,) let us take the moment of these forces about C;

$$\therefore\ R\,.\,DC - W\,.\,a\cos\theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Proceeding in a similar manner with the beam CB, we find

$$R'\,.\,EC - W'\,.\,a'\cos\phi = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Again, when the equilibrium is once established, we may suppose the hinge C to become rigid; under this hypothesis the rigid body ACB is kept in equilibrium by four forces, viz. R, R', W and W'. Hence resolving them vertically and horizontally, we find

$$R\cos\theta + R'\cos\phi - W - W' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\text{and } R\sin\theta - R'\sin\phi = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

These four are all the independent equations which can be derived from the mechanical properties of the machine; there are however two others, which express its geometrical properties, derived from the triangle DCE, viz.

$$(5) \ldots\ldots DC = \frac{b\sin\phi}{\sin(\theta+\phi)}, \text{ and } EC = \frac{b\sin\theta}{\sin(\theta+\phi)} \ldots\ldots\ldots (6).$$

From (1) and (5), we find

$$R = \frac{Wa\cos\theta\sin(\theta+\phi)}{b\sin\phi},$$

and from (2) and (6)

$$R' = \frac{W'a'\cos\phi\sin(\theta+\phi)}{b\sin\theta},$$

which being substituted in (3) and (4) give

$$W + W' = \sin(\theta+\phi)\left(\frac{Wa\cos^2\theta}{b\sin\phi} + \frac{W'a'\cos^2\phi}{b\sin\theta}\right),$$

$$\text{and } 0 = Wa\sin^2\theta\cos\theta - W'a'\sin^2\phi\cos\phi,$$

which two equations are sufficient for the determination of θ and ϕ.

Cor. If the rods are equal in all respects, these two equations become

$$\frac{2b}{a} = \sin(\theta + \phi)\left(\frac{\cos^2\theta}{\sin\phi} + \frac{\cos^2\phi}{\sin\theta}\right) \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\text{and } 0 = \sin^2\theta\cos\theta - \sin^2\phi\cos\phi;$$

the last of which gives

$$\theta = \phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A),$$

$$\text{or } 1 = \cos^2\theta + \cos\theta\cos\phi + \cos^2\phi \ldots\ldots\ldots\ldots\ldots (B).$$

Let us consider these two results separately, and

(1) When $\theta = \phi$ equation (7) gives

$$\cos\theta = \left(\frac{b}{2a}\right)^{\frac{1}{3}},$$

whence the position of the rods is known. This position is symmetrical with regard to a vertical line through C.

(2) The equations (B) and (7) shew that θ and ϕ are interchangeable, and consequently there are, besides the symmetrical position just found, two unsymmetrical positions of equilibrium, similarly situated on each side of the first found position. They may be found by means of (7) and (B).

28. *A solid of any form is placed with its convex surface upon a horizontal plane; to find the position of equilibrium.*

Let $z = f(x, y)$ be the equation of the surface, referred to three rectangular axes in the body: and let xyz be the co-ordinates of the point in contact with the horizontal plane, and $\bar{x}\,\bar{y}\,\bar{z}$ those of the centre of gravity referred to the same axes. Then the plane on which the body stands being a tangent plane, if $\alpha\beta\gamma$ be the inclinations of the co-ordinate axes to the horizon, $\frac{\pi}{2} - \alpha$, $\frac{\pi}{2} - \beta$, $\frac{\pi}{2} - \gamma$ will be the inclinations of the vertical through the point of contact to the co-ordinate axes; this vertical line is a normal, and therefore

$$\left.\begin{aligned}\sin\alpha &= \frac{-d_x z}{\sqrt{1+(d_x z)^2+(d_y z)^2}}\\ \sin\beta &= \frac{-d_y z}{\sqrt{1+(d_x z)^2+(d_y z)^2}}\\ \sin\gamma &= \frac{1}{\sqrt{1+(d_x z)^2+(d_y z)^2}}\end{aligned}\right\} \dots\dots\dots\dots\dots\dots(1).$$

But since the solid rests upon a point, the vertical through that point must pass through the centre of gravity of the solid, i. e. the normal at the point of contact passes through the centre of gravity of the solid; hence the equations of the normal give

$$\left.\begin{aligned}0 &= x-\bar{x}+d_x z\,.\,(z-\bar{z})\\ 0 &= y-\bar{y}+d_y z\,.\,(z-\bar{z})\end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots(2).$$

These two equations, together with the equation

$$z=f(x, y)$$

will enable us to find x, y, z; and thence α, β, γ from (1).

Ex. *Suppose the solid to be the eighth part of a sphere comprehended between three rectangular planes: to find the position in which it will rest with its convex surface on a horizontal plane.*

Let its equation be

$$a^2=x^2+y^2+z^2;$$

$$\therefore\ d_x z=-\frac{x}{z},\ \text{and}\ d_y z=-\frac{y}{z}.$$

Also from Ex. 1, Art. 177, we have

$$\bar{x}=\frac{3}{8}a,\quad \bar{y}=\frac{3}{8}a,\quad \bar{z}=\frac{3}{8}a,$$

hence making substitution in equations (2) we obtain

$$x=y=z;$$

these two equations joined with the equation of the surface of the sphere give

$$x=\frac{a}{\sqrt{3}},\quad y=\frac{a}{\sqrt{3}},\quad z=\frac{a}{\sqrt{3}};$$

$$\therefore\ d_x z=-1,\quad \text{and}\ d_y z=-1.$$

Consequently

$$\sin\alpha = \sin\beta = \sin\gamma = \frac{1}{\sqrt{3}}.$$

29. *To determine the nature of the equilibrium when a body of given form rests upon a given curve surface.*

At the point of contact of the given body with the surface on which it rests in equilibrium, the two surfaces will have a common normal, which will be vertical and pass through the centre of gravity of the body. Let DAd (fig. 104) be this normal, A being the point of contact of the two surfaces BC, bc; and D, d being the centres of curvature of the arcs BC, bc corresponding to the point A; and let G be the centre of gravity of the body. Let now the body be made to roll over a very small arc AP, and thereby to come into the position $b'c'$; A', G', d' being the new positions of A, G, d; and P being the new point of contact. By this movement the point A' will trace out a small portion of an epicycloid, which at the very beginning of the motion is perpendicular to the surface at A; hence A' begins to move along the line Ad. We suppose the displacement of the body so small that A' is in Ad. Draw Pp vertical. If Pp pass through G' the body is still in equilibrium; but if G' lie to the right of Pp (as in the figure), the body when left to itself will roll back into its original position; and lastly, if G' lie to the left of Pp, the body will roll farther from its first position. Hence the first position is one of *stable*, *unstable*, or *neuter* equilibrium according as

$$A'p \text{ is} > < \text{or} = A'G'.$$

To express this result analytically, let ρ, ρ' be the radii of curvature DA, dA.

Then because the lines Pp, DA' (for A' is in the line Dd) are parallel;

$$\therefore\ \frac{Dd'}{DP} = \frac{A'd'}{A'p} \text{ or } \frac{\rho+\rho'}{\rho} = \frac{\rho'}{A'p}.$$

$$\therefore\ A'p = \frac{\rho\rho'}{\rho+\rho'}.$$

Hence the equilibrium is *stable*, *unstable*, or *neuter*, according as

$$\frac{\rho\rho'}{\rho+\rho'} \text{ is } > < \text{ or } = AG.$$

COR. 1. If the surface on which the body rests be concave, we must account ρ negative in the above result.

COR. 2. If the surface be a plane, we must make ρ infinite, and then since

$$\frac{\rho\rho'}{\rho+\rho'} = \frac{\rho'}{1+\frac{\rho'}{\rho}} = \rho',$$

in that case, the equilibrium will be *stable*, *unstable*, or *neuter* according as

$$\rho' \text{ is } > < \text{ or } = AG.$$

COR. 3. If the lower surface of the body be a plane, we must make ρ' infinite, and then the result is

$$\rho \text{ is } > < \text{ or } = AG.$$

Ex. *Find what segment of a paraboloid will rest in a position of neuter equilibrium upon a spherical surface of given radius.*

Let x be the length of the axis of the paraboloid, $4m$ its latus rectum; and a the radius of the spherical surface. Then from Ex. 5, p. 101, we have

$$AG = \frac{2x}{3},$$

and by the *Differential Calculus*

$$\rho' = 2m,$$

$$\text{and } \therefore \frac{2x}{3} = \frac{2ma}{2m+a};$$

$$\therefore x = \frac{3ma}{2m+a}.$$

30. *A string is stretched along a smooth curve line of any form by two equal forces, required the unit of pressure exerted by it upon the cylinder at any point.* (Fig. 105.)

Let AHB be the curve line along which the string is stretched by the two equal forces P, Q. Let HH' be a very small arc, and at H, H' draw tangents meeting in K, and normals HO, $H'O$. Join KO, and put $\angle HOH' = \delta\theta$. The portion of string HH' is kept in equilibrium by the tensions at H, H', each of which is equal to P or Q; and by the reactions of the curve line HH', which being smooth, the reaction at every point will be in the direction of a normal. Hence the resultant of all the reactions on HH' will pass through O, and as it must also pass through K, it acts in the line OK. Hence by Art. 28,

$$\begin{aligned} \text{resultant reaction on } HH' : P &:: \sin HKH' : \sin OKH \\ &:: \sin HOH' : \cos KOH \\ &:: \delta\theta : 1 \text{ ultimately}; \end{aligned}$$

$$\therefore \text{ resultant reaction on } HH' = P.\,\delta\theta.$$

Now when the arc HH' is diminished indefinitely, the pressures upon it may be all considered parallel, and therefore their resultant is equal to their sum.

Consequently the pressure upon the indefinitely small arc HH' is equal to $P\delta\theta$ or to

$$P.\frac{\text{arc}}{\text{rad. curv.}},$$

and the unit of pressure (or pressure on an arc of the length unity)

$$= \frac{\text{tension of string}}{\text{rad. curvature}}.$$

Cor. Let C, D be the points where the string leaves the arc; and let p be the whole pressure upon CH; and let $\theta =$ the angle between the normal at C and that at H; then $\delta p =$ the pressure upon HH', and by what has just been proved

$$\delta p = P\delta\theta;$$

$$\therefore\ p + C = P\theta.$$

But when $\theta = 0$, $p = 0$, and therefore $C = 0$;

$$\therefore\ p = P\theta.$$

If α be the angle included between the normals at C and D, and p' the pressure upon the whole arc CHD,

$$p' = P\alpha.$$

It is remarkable that this result is independent of the form of the curve, provided it be in every part convex towards the string in contact with it.

31. *A string is stretched along a rough curve line of any form by two such forces that the string is on the point of moving. Having given the coefficient of friction, find the proportion of the forces.* (Fig. 105.)

Let Q be the larger force; t, $t + \delta t$ the tensions of the string at H, H'; μ the coefficient of friction; θ, $\delta\theta$, α as before. Then the pressure on $HH' = t\delta\theta$, and therefore the friction on $HH' = \mu t\delta\theta$; but the arc HH' being pulled in opposite directions by the forces t, $t + \delta t$, the latter is prevented from producing motion only by the friction on HH';

$$\therefore\ t + \delta t - t = \mu t\delta\theta;$$

$$\therefore\ \frac{d_\theta t}{t} = \mu;$$

$$\therefore\ \log_e t = \mu\theta + C,$$

when $\theta = 0$, $t = P$, and when $\theta = \alpha$, $t = Q$; wherefore

$$\log_e Q - \log_e P = \mu\alpha;$$

$$\therefore\ Q = Pe^{\mu\alpha}.$$

This result is independent of the form of the curve.

32. *A uniform heavy chain is laid upon a smooth arc of a quadrant of a circle, and coincides with it; one of the bounding radii of the quadrant being horizontal, and the other vertical. Find the force necessary to prevent the chain from sliding down the arc: and compare the pressure upon the curve with the weight of the chain.* (Fig. 106.)

Let F be the force which, acting horizontally at B, just prevents the chain from sliding down the quadrant. Let P, Q be two points very near to each other; $a = AO$, $\theta = AOP$, $\delta\theta = POQ$, t and $t + \delta t$ the tensions of the chain at P and Q; $p =$ the pressure upon the arc AP, and $\delta p =$ that upon PQ; $\rho =$ the weight of a piece of the chain of the length 1. Then the elementary portion of chain PQ is kept in equilibrium by the tensions t, $t + \delta t$, its own weight $\rho a \delta\theta$, and the reactions of the curve PQ; to avoid the latter, take the moments of these forces about the point O;

$$\therefore\ 0 = (t + \delta t)\,a - ta - \rho a \delta\theta \,.\, a \cos\theta;$$

$$\therefore\ d_\theta t = \rho a \cos\theta;$$

$$\therefore\ t = \rho a \sin\theta + C.$$

But at A, $t = 0$, $\theta = 0$, and $\therefore\ C = 0$;

$$\therefore\ t = \rho a \sin\theta.$$

And at B, $t = F$ and $\theta = \frac{\pi}{2}$;

$$\therefore\ F = \rho a$$

$=$ the weight of a piece of the chain, the length of which is equal to the radius.

Again, to find the pressure upon the quadrant.

The pressure of the elementary portion PQ is due to two causes, viz. its own weight, and the tensions t and $t + \delta t$. The former part $= \rho a \delta\theta \sin\theta$, and the latter part $= t\delta\theta$, by Prob. 29;

$$\therefore\ \delta p = \rho a \delta\theta \sin\theta + t\delta\theta;$$

$$\therefore\ d_\theta p = \rho a \sin\theta + \rho a \sin\theta;$$

$$\therefore\ p = -2\rho a \cos\theta + C.$$

At A, $\theta = 0$ and $p = 0$; $\therefore\ C = 2\rho a$;

$$\therefore\ p = 2\rho a\,(1 - \cos\theta),$$

and at B, $\theta = \frac{\pi}{2}$ and $p =$ the whole pressure on the quadrant,

$$= 2\rho a;$$

$$\therefore \frac{\text{press. on quad.}}{\text{weight of chain}} = \frac{2\rho a}{\rho a \frac{\pi}{2}} = \frac{4}{\pi}.$$

33. *Supposing the quadrant to be rough, to find the least value of* F *which can prevent the chain from sliding off; having given the coefficient of friction* $(= \mu)$.

In this case the chain is on the point of moving towards the point A, consequently friction acts up the quadrant.

The forces which keep PQ in equilibrium are t, $t + \delta t$, $\mu \delta p$, $\rho a \delta \theta$, and the normal reactions; to avoid the last, take the moments of the forces about the point O;

$$\therefore\ 0 = (t + \delta t)\, a - ta + \mu \delta p a - \rho a \delta \theta \,.\, a \cos \theta;$$

$$\therefore\ d_\theta t + \mu d_\theta p = \rho a \cos \theta \dots\dots\dots\dots\dots\dots (1).$$

Also as before

$$\delta p = \rho a \delta \theta \,.\, \sin \theta + t \delta \theta;$$

$$\therefore\ d_\theta p = \rho a \sin \theta + t.$$

Hence substituting this value of $d_\theta p$ in (1), we obtain

$$d_\theta t + \mu\, (\rho a \sin \theta + t) = \rho a \cos \theta;$$

$$\therefore\ d_\theta t + \mu t = \rho a\, (\cos \theta - \mu \sin \theta).$$

Multiplying this equation by $e^{\mu\theta}$ and integrating, we find

$$t e^{\mu\theta} = \frac{2\mu \cos \theta + (1 - \mu^2) \sin \theta}{1 + \mu^2} \,.\, \rho a e^{\mu\theta} + C.$$

Now when $\theta = 0$, $t = 0$;

$$\therefore\ 0 = \frac{2\mu}{1 + \mu^2} \,.\, \rho a + C.$$

And when $\theta = \frac{\pi}{2}$, $t = F$;

$$\therefore \; Fe^{\frac{\mu\pi}{2}} = \frac{1-\mu^2}{1+\mu^2} \cdot \rho a e^{\frac{\mu\pi}{2}} + C$$

$$= \frac{1-\mu^2}{1+\mu^2} \cdot \rho a e^{\frac{\mu\pi}{2}} - \frac{2\mu}{1+\mu^2}\rho a\,;$$

$$\therefore \; F = \frac{1-\mu^2}{1+\mu^2} \cdot \rho a - \frac{2\mu\rho a}{1+\mu^2} \cdot e^{-\frac{\mu\pi}{2}}.$$

COR. If the pressure be required, it may be found by integrating equation (1);

$$\therefore \; t + \mu p = \rho a \sin\theta,$$

no constant being added, because t, p, θ vanish together;

$$\therefore \; p = \frac{\rho a}{\mu} \cdot \sin\theta - \frac{t}{\mu}.$$

Hence when $\theta = \frac{\pi}{2}$, we have the pressure on the quadrant

$$= \frac{\rho a}{\mu} - \frac{F}{\mu}$$

$$= \frac{\rho a}{\mu} - \frac{1-\mu^2}{1+\mu^2} \cdot \frac{\rho a}{\mu} + \frac{2\rho a}{1+\mu^2} \cdot e^{-\frac{\mu\pi}{2}}$$

$$= \frac{2\rho a}{1+\mu^2} \cdot \left(\mu + e^{-\frac{\mu\pi}{2}}\right).$$

APPENDIX.

ON THE COMPOSITION OF TWO FORCES ACTING ON A POINT.

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1. SINCE two forces which are in equilibrium must necessarily be equal and opposite, two forces F_1 and F_2 which do not act in opposite directions, must necessarily have a resultant, the position of which we shall proceed to determine.

(1) *The resultant of two forces* F_1 *and* F_2 *acting on a point* P, *is situated in the plane* $\mathrm{F}_1\mathrm{PF}_2$.

For if it be not in that plane, it must be either above or below. But it cannot be above; for, any reason which would assign it such a position might be used to assign it a similar position below; for these two positions are similarly situated with regard to the forces F_1 and F_2; there would consequently be two resultants, which is impossible. The resultant then cannot be situated above the plane of the forces; and in a similar way we may shew that it cannot be situated below, and therefore it must be in the plane.

(2) *It lies within the angle* $\mathrm{F}_1\mathrm{PF}_2$.

For the tendency of F_1 is to draw the particle P in the direction PF_1, while that of F_2 is to draw it in the direction PF_2, and hence the real motion, which is the result of these united tendencies, will not be in the direction of either, but intermediate to both; and therefore within the angle F_1PF_2: consequently the resultant, which is a single force that would produce the same motion, must be situated within the angle F_1PF_2.

2. Since F_1 and F_2 do in part hinder each other from producing their whole effects, it appears that their resultant must be less than their sum; for their resultant can only be equal to their

sum when neither interferes with the other, which is not the case unless they act in the same direction; consequently

$$R < F_1 + F_2.$$

3. *If the forces* F_1 *and* F_2 *are equal, their resultant* R *will bisect the angle* F_1PF_2.

For if there be a reason why PR should lie nearer to PF_1 than to PF_2, there must be a similar reason why it should lie nearer to PF_2 than to PF_1, since the forces are equal; and hence there would be two resultants, which is impossible; consequently PR bisects the angle F_1PF_2.

4. Having thus determined the direction of the resultant of two equal forces, we proceed *to find its magnitude.*

Let F_1, f_1 (fig. 107) be two equal forces acting on the particle P, and R their resultant bisecting the angle F_1Pf_1. Since R is less than the sum of the two forces F_1 and f_1 it is clear that $\frac{R}{F_1+f_2}$, or its equal $\frac{R}{2F_1}$, is always less than 1; and, consequently, an angle θ may be found such that

$$\frac{R}{2F_1} = \cos\theta,$$

$$\text{or } R = 2F_1 \cos\theta.$$

The angle θ is unknown at present, but from Art. 19 we learn that so long as the angle F_1Pf_1 remains the same, θ continues unchanged; that is, if we have two sets of forces inclined at the same angle with each other respectively, we shall have $R = 2F_1 \cos\theta$, and $R' = 2F'_1 \cos\theta$, and therefore

$$R : R' :: F_1 : F_1' \ldots\ldots\ldots\ldots\ldots\ldots\ldots(A),$$

that is, the resultants are proportional to the components.

Let now F_2, f_2 be two other equal forces acting on P whose resultant is also equal to R, the angles F_1PF_2, f_1Pf_2 being each equal to RPF_1 or RPf_1. Now at P apply four forces, each equal to x, two of them respectively in the directions PF_2, Pf_2, and the other two in the direction PR; and let them be of such magnitude, that F_1 may be the resultant of the one in the direction PF_2 and one in the direction PR. Then, since these two contain the same angle as F_1 and f_1, and F_1 is their resultant,

$$F_1 = 2x \cos \theta.$$

Also, if we substitute instead of F_1 and f_1, their components, we may consider R as the resultant of the forces x, x, x, and x; of which two act in the same direction as R; and, consequently, $R - 2x$ is the resultant of the two x, x, which act in the directions PF_2, Pf_2; and since, by hypothesis, R is the resultant of F_2 and f_2, which act in the same directions as x, x,

$$\therefore\ R : R - 2x :: F_2 : x, \quad \text{from (A)};$$

$$\therefore\ \frac{R}{F_2} = \frac{R}{x} - 2.$$

But $R = 2F_1 \cos \theta = 2 \,.\, 2x \cos \theta \,.\, \cos \theta = 4x \cos^2 \theta$;

$$\therefore\ \frac{R}{F_2} = 4 \cos^2 \theta - 2$$

$$= 2 (2 \cos^2 \theta - 1)$$

$$= 2 \cos 2\theta;$$

$$\therefore\ R = 2F_2 \cos 2\theta.$$

It appears then, that in the formula

$$R = 2F_1 \cos \theta,$$

if we double the angle at which the forces are inclined, we must also double θ.

We will now suppose, that when the angle at which the forces act is a multiple n, or any inferior multiple, of F_1Pf_1, it is true that in the same formula the corresponding equimultiple of θ is to be taken; so that

$$R = 2F_n \cos n\theta = 2F_{n-1} \cos (n-1)\, \theta = \ldots = 2F_1 \cos \theta.$$

Apply (fig. 108) at P, as before, four forces in the directions PF_{n+1}, PF_{n-1}, Pf_{n+1}, and Pf_{n-1} respectively, each of such a magnitude x that F may be the resultant of the two in the directions PF_{n+1}, PF_{n-1}, and f_n of the other two;

$$\therefore F_n = 2x \cos\theta.*$$

But if, instead of the forces F_n, f_n, we substitute their four components, we may consider R as the resultant of the forces x, x, x, and x, of which two acting in the directions PF_{n-1}, Pf_{n-1} will have $2x\cos(n-1)\theta$ for their resultant in the direction PR, and consequently $R - 2x\cos(n-1)\theta$ is the resultant of the other two which act in the same directions as F_{n+1} and f_{n+1}; consequently, from (A),

$$R : R - 2x\cos(n-1)\theta :: F_{n+1} : x;$$

$$\therefore \frac{R}{F_{n+1}} = \frac{R}{x} - 2\cos(n-1)\theta$$

$$= \frac{R}{x} - 2\cos\theta\cos n\theta - 2\sin\theta\sin n\theta.$$

But $\quad R = 2F_n\cos n\theta = 4x\cos\theta\cos n\theta$;

$$\therefore \frac{R}{F_{n+1}} = 4\cos\theta\cos n\theta - 2\cos\theta\cos n\theta - 2\sin\theta\sin n\theta$$

$$= 2(\cos\theta\cos n\theta - \sin\theta\sin n\theta)$$

$$= 2\cos(n+1)\theta;$$

$$\therefore R = 2F_{n+1}\cos(n+1)\theta.$$

Hence the formula is true for a multiple $(n+1)$ if it be true for n and all inferior multiples: but it has been shewn to be true for 2 and 1, and consequently it is true for multiples 3, 4, 5, 6, ... and generally, by induction, for any multiple whatever.

It appears then, that as we increase the angle at which two equal forces (F, f) act, we must increase the angle θ in the same proportion, and then, that the formula

$$R = 2F\cos\theta$$

* For the $\angle F_{n+1}PF_{n-1} = \angle F_1 Pf_1$, (fig. 107).

still holds good. This, however, supposes the angle between the forces to be a multiple of F_1Pf_1 (fig. 107), which may not happen to be the case; but by taking the original angle F_1Pf_1 exceedingly small, we may find a multiple of it which shall differ from FPf a proposed angle by less than any assignable quantity. It is evident then, that FPf and θ have an invariable ratio to each other, so that if $FPf = 2\phi$, then

$$\frac{\theta}{\phi} = \text{constant quantity} = c \text{ suppose};$$

$$\therefore\ R = 2F\cos c\phi.$$

To determine the value of c, we observe that if F and f act at an angle π, or are opposite to each other, $\left(\text{in which case } \phi = \frac{\pi}{2}\right)$ they have no resultant;

$$\therefore\ 0 = 2F\cos\frac{c\pi}{2},$$

$$\therefore\ \cos\frac{c\pi}{2} = 0.$$

Now none but angles which are odd multiples of $\frac{\pi}{2}$ have their cosines $= 0$;

$$\therefore\ c = \text{an odd integer}, = 1 \text{ as we shall shew.}$$

For if c is not $= 1$, let the angle FPf be such that $\phi = \frac{\pi}{2c}$, which is therefore less than a right angle, and then

$$R = 2F\cos c\phi = 2F\cos\frac{\pi}{2} = 0.$$

But since the angle FPf is, in this case, $= \frac{\pi}{c}$, and therefore less than π, the resultant cannot be $= 0$, which is absurd, and consequently $c = 1$. We arrive therefore at the general result, that if F, f be two equal forces acting on a particle, and inclined to each other at the angle 2ϕ, their resultant R is inclined to each of them at the angle ϕ, and its magnitude is determined by the equation

$$R = 2F\cos\phi.$$

5. It will be immediately obvious that, since the forces F and f are perfectly equal and similarly situated with respect to PR, they contribute equally to the resultant R; and, consequently the efficiency of each in the direction PR is equal to $\frac{1}{2}R$, or $F\cos\phi$.

6. *To determine the magnitude and direction of the resultant of any two forces acting on a particle.*

Let F, f (fig. 109) be the two forces acting on the particle P; R their resultant, perpendicular to which draw LPM; let α, β denote the angles FPR, fPR respectively, and ϕ the angle FPf between the forces. Then the efficiencies of F and f, in the direction PR, are respectively $F\cos\alpha$, $f\cos\beta$, the sum of which must be equal to R, since the efficiency of R is equivalent to the united efficiencies of F and f in any proposed direction, because R is their resultant;

$$\therefore\ R = F\cos\alpha + f\cos\beta \ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Now the efficiency of R in the direction PL perpendicular to itself $= R\cos 90^\circ = 0$; and the efficiency of F in the direction $PL = F\cos FPL$, and that of f in the same direction $= f\cos fPL$;

$$\therefore\ 0 = F\cos FPL + f\cos fPL,$$

$$\text{or}\ \ 0 = F\cos\left(\frac{\pi}{2} - \alpha\right) + f\cos\left(\frac{\pi}{2} + \beta\right),$$

$$\text{or}\ \ 0 = F\sin\alpha - f\sin\beta \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

and by squaring equations (1) and (2), we have

$$R^2 = F^2\cos^2\alpha + 2Ff\cos\alpha\cos\beta + f^2\cos^2\beta,$$

$$0 = F^2\sin^2\alpha - 2Ff\sin\alpha\sin\beta + f^2\sin^2\beta;$$

and adding these together,

$$R^2 = F^2 + 2Ff(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + f^2.$$

But because $\phi = \alpha + \beta$;

$\therefore$ $\cos\phi = \cos\alpha\cos\beta - \sin\alpha\sin\beta$; and, consequently,

$$R^2 = F^2 + 2Ff\cos\phi + f^2.$$

This equation shews that the diagonal of a parallelogram represents the *magnitude* of the resultant of two forces, which are themselves represented in magnitude and direction by the sides: and equation (2) shews that the same diagonal also represents the *direction* of the resultant.

MISCELLANEOUS PROBLEMS.

1. Two given weights are suspended from the ends of a bent lever, the arms of which are given, and include a given angle; find the position of equilibrium.

2. A bent lever of uniform thickness rests with its shorter arm horizontal. But if the length of this arm were doubled the lever would rest with the other arm horizontal. Compare the lengths of the arms, and find their inclination.

3. Two forces act at angles $\alpha\beta$ upon the arms a, b of a straight lever which is not attached to its fulcrum. Shew that if there be equilibrium $a : b :: \tan\beta : \tan\alpha$.

4. The beam of a false balance being uniform, shew that the lengths of the arms are respectively proportional to the differences between the true and apparent weights of a given substance.

5. A beam of oak 30 feet long balances upon a point 10 feet from one end: but when a weight of 10 lbs. is suspended at the thin end, the prop must be moved 2 feet to preserve equilibrium. Find the weight of the oak.

6. Two equal forces act in opposite directions along two opposite sides of a parallelogram, and a third force along the diagonal. Find the force which will keep them in equilibrium.

7. If forces proportional to the sides of a polygon be applied in the plane of the figure at the middle points of the sides and perpendicular thereto, they will balance.

8. A given body is supported on an inclined plane, first by a power parallel to the base, and then by a power parallel to the plane. Compare the pressures on the plane in the two cases.

9. A rope of given length is used to pull down a vertical pillar; at what height from the base of the pillar must it be fastened that a given force pulling it may be most efficacious?

10. A weight P hangs vertically by a string from a fixed point A; a string PBW being now fastened to P is passed over a fixed pulley B (so that BP is horizontal) and supports a weight W. Find how much this will draw AP from the vertical.

11. C, D are two smooth pegs, and $ACDB$ is a heavy circular arc, which passes over one peg and under the other; find the position of equilibrium.

12. A given sphere rests between two given inclined planes; find the pressure upon each.

13. Two weights support each other on two given inclined planes which have a common vertex, by means of a string passing over the vertex; find the proportion of the weights.

14. A given cone is placed with its base on an inclined plane, the coefficient of friction for which is known: determine whether, upon increasing the inclination of the plane, the cone will tumble or slide.

15. A weight is suspended from one extremity of a string which passes over two fixed pulleys and through a ring at its other extremity; find the position of equilibrium.

16. A given beam rests with its lower end on a smooth horizontal plane, and its upper end on a given inclined plane; find the force which must act at the foot of the beam to prevent sliding.

17. Two given heavy particles being connected by an inflexible rod of given length are placed within a hemispherical bowl; find the position of equilibrium, and the compressing force upon the rod.

18. A rigid rod AB is moveable in a vertical plane about a fixed hinge A, the end B leans against a smooth vertical wall. Find the pressures on the wall and hinge.

19. A beam of given length and weight is placed with one end on a vertical, and the other on a horizontal plane; find the force necessary to keep it at rest, and the pressures on the two planes.

20. A and B are two given points in a horizontal line, to which are fastened two strings AC, BCW of given lengths; the string BCW passes through a ring attached to the string AC, and to it is fastened a given weight W; find the position in which the ring will rest.

21. AC, BCD are two given beams moveable in a vertical plane about hinges A, B in a horizontal plane. BD the longer leans upon the end C of AC the shorter. Find the position of equilibrium.

22. If a rod rest in equilibrium with its ends on two smooth inclined planes, the intersection of the planes must be a horizontal line.

23. A beam has a ring at one extremity which moves up and down a vertical rod. Find the position of the beam when it rests upon the arc of a circle a diameter of which coincides with the rod.

24. The upper end of a given rod rests against a smooth vertical plane, and the lower end is suspended by a given string fastened to a point in the plane; find the position of equilibrium.

25. A given uniform rod passing freely through an orifice in a vertical plane rests in equilibrium with one end upon a given inclined plane; find its position.

26. A heavy beam leans against an upright prop; the lower end of the beam rests upon the horizontal plane and is prevented from sliding by a string tied to the bottom of the prop; required the tension of this string.

27. Out of a square it is required to cut a triangle such that the remaining figure may have its centre of gravity where the vertex of the triangle was.

28. If the sides of a triangle taken in order be cut proportionally, the triangle formed by joining the points of division will have the same centre of gravity as the original triangle.

29. Find the distance of the centre of gravity from the angular point of a uniform bent lever whose arms and the angle which they include are given.

30. A solid composed of a cone and a hemisphere of equal bases, placed base to base, rests with the convex surface of the hemisphere upon a horizontal plane, and the axis of the cone in an inclined position; compare the dimensions of the cone and hemisphere.

31. Determine the point in its curve surface on which a semi-paraboloid will rest on a horizontal plane.

32. A solid generated by the revolution of a quadrant of an ellipse about its major axis, is placed upon a horizontal plane, with its axis in an oblique position; determine the position of equilibrium.

33. An ellipsoid rests on a horizontal plane on the extremity of its mean axis; shew how to estimate the stability with regard to a slight displacement in any direction. Define the direction which distinguishes between stable and unstable equilibrium.

34. The centre of gravity of three weights $a \,.\, (w-\alpha)^2$, $b \,.\, (w-\beta)^2$, $c \,.\, (w-\gamma)^2$, whatever be the value of w, will be situated in a line of the second order to which the lines joining the centres of gravity of the weights are tangents.

35. If a hemisphere and paraboloid of equal bases and similar materials have their bases cemented together, the whole solid will rest on a horizontal plane on any point of the spherical surface if the altitude of the paraboloid : the radius of the hemisphere :: $\sqrt{3} : \sqrt{2}$.

36. Three uniform beams AB, BC, CD, of the same thickness, and of lengths l, $2l$, l respectively, are connected by hinges at B and C, and rest on a perfectly smooth sphere, the radius

of which $=2l$, so that the middle point of BC, and the extremities A, D are in contact with the sphere; shew that the pressure at the middle point of $BC=\frac{91}{100}$ of the weight of the beams.

37. A sphere of given weight and radius is suspended by a string of given length from a fixed point, to which point also is attached another given weight by a string so long that the weight hangs below the sphere; find the angle which the string, to which the sphere is attached, makes with the vertical.

38. Two given beams AC, BDC lean against each other in a vertical plane; and their ends A, B resting on a smooth horizontal plane are prevented from sliding by a string AD, which is fastened to the beam AC at A, and the beam BDC at D. Find the tension of the string.

39. A cylinder, with its base resting against a smooth vertical plane, is held up by a string fastened to it at a point of its curved surface whose distance from the vertical plane is h. Shew that h must be $> b-2a\tan\theta$ and $< b$, where $2b$ is the altitude of the cylinder, a the radius of the base, and θ the angle which the string makes with the vertical.

40. A flat board in the form of a square is supported upon two props with its plane vertical; determine its positions of equilibrium, friction being neglected, and the distance between the props being equal to half a side of the square.

41. Determine the position of equilibrium of a uniform rod, one end of which rests against a plane perpendicular to the horizon, and the other on the interior surface of a given hemisphere.

42. If the sides of a triangle ABC be bisected in the points D, E, F; then the centre of the circle inscribed in the triangle DEF is the centre of gravity of the perimeter of the triangle ABC.

43. Three equal rods, loosely connected together by one extremity of each, have their other extremities placed upon a

rough horizontal plane at the angular points of a given equilateral triangle. A smooth heavy ring is then placed on the rods; find the coefficient of friction between the rods and the plane that the machine may just be on the point of falling.

44. A cone and sphere of given weights support each other between two given inclined planes, the cone resting on its base on one of the planes; determine what must be the vertical angle of the cone, that the equilibrium may subsist.

45. A given cylinder with its axis horizontal is held at rest on a given rough inclined plane by a string coiled round its middle and then fastened to the top of the plane; find the position of equilibrium.

46. A given weight is placed upon a rough horizontal plane; required the magnitude and direction of the least force which will be able to move it.

47. The resultant and sum of two forces are given, and also the angle which one of them makes with the resultant; it is required to determine the forces and the angle at which they act.

48. A circular hoop is supported in a horizontal position, and three weights of 4, 5, and 6 pounds respectively are suspended over its circumference by three strings fastened together in a knot. Shew that the knot must be in the centre of the hoop, and find what must be the positions of the strings so that they may sustain one another.

49. Four beams, AB, BC, CD, DA (fig. 27) connected by hinge joints, have the opposite corners connected by two elastic strings AC, BD. Shew that

$$\text{tension of } AC : \text{tension of } BD :: \frac{AE \,.\, EC}{AC} : \frac{BE \,.\, ED}{BD}.$$

50. A uniform straight rod rests with its middle point upon a rough horizontal cylinder, their directions being horizontal and perpendicular to each other. Find the greatest weight which may be attached to one end of the rod without causing it to slide off the cylinder.

51. Two equal uniform beams connected by a hinge joint are laid across a smooth horizontal cylinder of given radius. Find their inclination to each other when in equilibrium.

52. A particle is placed on the surface of an ellipsoid, and is attracted towards the principal planes by forces which are respectively proportional to its distance from them; find the conditions of equilibrium.

53. Prove the equality of the power and weight in Roberval's balance by couples; and find the strains upon the joints and pins.

54. A particle is placed on the arc of a given parabola, and is acted on by gravity parallel to the axis, and a force perpendicular to it which is proportional to the distance of the particle from the axis; find the position of equilibrium.

55. If three parallel forces acting at the angular points A, B, C of a plane triangle are respectively proportional to the opposite sides a, b, c; prove that the distance of the centre of parallel forces from A

$$=\frac{2bc}{a+b+c}\cos\frac{A}{2}.$$

56. A ladder rests with its foot on a horizontal plane, and its upper extremity against a vertical wall; having given its length, the place of its centre of gravity, and the ratios of the friction to the pressure both on the plane and on the wall; find its position when in a state bordering upon motion.

57. If a lever, kept at rest by weights P, Q, suspended from its arms a, b, so that they make angles α, β, with the horizon, be turned about its fulcrum through an angle 2θ, prove that the vertical spaces described by P and Q, are to one another as $a\cos(\alpha+\theta) : b\cos(\beta-\theta)$; and thence deduce the equation of virtual velocities.

58. If S and D represent respectively the semi-sum and semi-difference of the greatest and least angles, which the direction of a power supporting a weight on a rough inclined plane

may make with the plane, and ϕ be the least elevation of the plane when a body would slide down it; prove that the cosine of the angle, at which the same power being inclined to a smooth plane of the same elevation would support the same weight,

$$=\frac{\cos S}{\cos \phi} \cdot \cos (D+\phi).$$

59. A roof ACB consisting of beams which form an isosceles triangle with its base AB horizontal, supports a given weight at C; find the horizontal force at A. Why must a pointed arch carry a heavy weight at its vertex?

60. Four equal uniform beams connected by joints are symmetrically placed in a vertical plane, in the form of a roof; shew that if the extremities be in a horizontal line, and θ, ϕ be the inclinations of the beams to the horizon, $\tan \theta = 3 \tan \phi$.

61. A beam AB, capable of motion in every direction about a fixed ball and socket joint at A, rests with its end B against a rough vertical plane; determine the extreme positions of equilibrium.

62. In the last question suppose the end B rests against a rough inclined plane; determine the extreme positions of equilibrium.

63. Three weights are suspended from the angles of an isosceles triangle, whose plane is vertical and which is supported by a horizontal axis, passing through its centre of gravity, about which it is capable of revolving. Determine its positions of equilibrium, the two weights suspended from the extremities of the base being equal, and each greater than the third: and shew in each case whether the equilibrium will be stable, unstable, or neutral.

64. A uniform rod, whose length is a, moveable freely in a vertical plane about a hinge at one extremity, is attracted by a force varying as D^{-2}, and acting from a centre at a height a directly above the hinge; find the position in which it will rest, and the nature of the equilibrium, supposing that the attractive force on the hinge is $\frac{1}{2}g$.

65. A hollow cylinder stands upon a smooth horizontal plane, and a light rod of given length, being in the same vertical plane with the axis of the cylinder, passes over the upper edge and rests against the interior surface. A given weight is attached to the other extremity of the rod, and the cylinder is just on the point of turning over. Determine its weight.

66. A cylinder is laid upon two equal cylinders all in parallel positions, and the lower ones resting in contact with each other upon a rough horizontal plane; find the relation between the coefficients of friction between the cylinders, and the coefficient of friction between a cylinder and the plane, that all the points of contact may begin to slip at the same instant.

67. Determine the conditions of equilibrium of a material point situated in an indefinitely thin bent tube of any form and acted upon by any number of forces.

68. A chain of uniform density is suspended at its extremities by means of two tacks in the same horizontal line at a given distance from each other; find the length of the chain so that the stress upon either tack may be equal to the chain's weight.

69. A uniform chain is suspended from two tacks in the same horizontal line at a given distance from each other. Find the length of the chain that the stress on the tacks may be the least possible.

70. A cylinder rests with the centre of its base in contact with the highest point of a fixed sphere, and four times the altitude of the cylinder is equal to a great circle of the sphere; supposing the surfaces in contact to be rough enough to prevent sliding in all cases, shew that the cylinder may be made to rock through an angle 90^0, but not more, without falling.

71. A man runs round in the circumference of a given circle with a given velocity; determine the inclination of his body to the horizon.

72. One end of a heavy rod can turn in every direction about a fixed point. The other end rests on the upper surface of a rough plane, (coefficient of friction μ) which is inclined to the horizon at an angle α. If β be the angle which the beam makes with the plane, prove that the rod will not rest in every position, unless $\cot^2\alpha$ be not less than $\frac{1}{\mu^2} + \tan^2\beta$.

73. A chain suspended at its extremities from two tacks in the same horizontal line forms itself into a cycloid; prove that the density at any point $\propto \sec^3(\frac{1}{2}\theta)$, and the weight of the corresponding arc $\propto \tan(\frac{1}{2}\theta)$, θ being the arc of the generating circle measured from the vertex.

74. A weight W is suspended from a point P of a uniform catenary APA'. O and O' are the lowest points of two uniform catenaries, of which AP and $A'P$ are parts. Shew that W is equal to the *difference* or *sum* of the weights of the portions OP, $O'P$ of the catenaries, according as AP and $A'P$ are *one* or *both* less than a semi-catenary.

75. If a chain acted on by gravity hang in the form of the curve whose equation is $\sec\frac{y}{a} = e^{\frac{x}{a}}$, shew that at every point the density or thickness is proportional to the tension.

76. A uniform catenary of given length is suspended from two given points at the same height, and is nearly horizontal; in consequence of an expansion of its materials the vertex of the catenary is observed to have descended through a small given altitude; find the increase of the length of the catenary, supposing its expansion to have been uniform throughout.

77. A uniform elastic string being of such a length that when it hangs vertically, if an equal quantity were appended to the lowest point it would stretch it to twice that length; what weight must be appended at the middle point that the increase of length may be three quarters of the original?

78. A given heavy elastic ring is passed over the vertex of a smooth vertical cone, and descends by its own weight; required the position of equilibrium.

79. A uniform heavy elastic string (natural length a) is stretched by forces applied at its ends, and then, being laid upon a rough inclined plane, is suffered slowly to contract itself. Shew that a point of the string, the natural distance of which from the upper end is

$$\frac{a}{2}\left(1+\frac{\tan\alpha}{\mu}\right)$$

will not be affected by friction. α is the inclination of the plane.

80. Find the form of a uniform chain suspended from any two points on the surface of an upright cone, and resting on the curve surface. Find the tension when it becomes a horizontal circle.

81. A given uniform rod is placed within a given rough hemispherical bowl; find the limiting positions of equilibrium.

82. If a right-angled triangle be supported in a horizontal position by vertical threads fastened to its angular points, each of which can just bear an additional tension of 1 lb., determine within what portion of the area a weight less than 3 lbs. may be placed without destroying the equilibrium.

83. Find the magnitude of the horizontal strain which a door exerts on its hinges; shew that the vertical strain on each hinge is indeterminate.

84. A beam, having one end on a vertical, and the other on a horizontal plane, is kept at rest by a string connecting its centre of gravity with the intersection of the planes. Find the tension of the string; and explain the result when the beam is uniform.

85. A cycloid is placed with its axis vertical; a weight is supported upon its arc by an elastic string, the natural length of which is given, and one end of which is fastened to the top of the cycloid; find the position of equilibrium.

86. Three equal spheres are placed in contact upon a rough horizontal plane. If another equal sphere, placed upon them, just causes them to separate, what is the coefficient of friction?

87. An elastic chain is laid upon a smooth inclined plane, one end being made fast to the top of the plane. The natural length of the chain is equal to the length of the plane; find how much of the chain will hang down off the plane when there is equilibrium.

88. A string binds tightly together two smooth cylinders of given radii. Compare the mutual pressure between the cylinders with the whole pressure of the string upon them.

89. Three equal smooth spheres are placed in mutual contact on a smooth horizontal plane, and are bound together by an elastic string in a plane containing their centres, the string not being stretched; another equal sphere is then placed upon them, and sinks till its lowest point is on a level with their centres. Find the elasticity of the string.

90. A string passing underneath a heavy pulley has its ends fastened to two points in a horizontal plane, the distance between the points being equal to the diameter of the pulley. Suppose the string to become elastic, and the pulley to be rough, find how far the pulley will sink below its first position.

91. When any number of forces act on a body, shew that the plane on which the sum of the projections of the *moments* of the forces about a fixed point is a maximum, is perpendicular to the plane with respect to which this sum is 0.

92. Assuming that if δp, δq, δr be the virtual velocities of three forces P, Q, R which keep a point at rest,

$$P\delta p + Q\delta q + R\delta r = 0,$$

in whatever direction the virtual motion of the point takes place; prove that the forces are proportional to the sides of a triangle drawn in their directions.

93. If A, B, C represent the moments of a force round each of three rectangular axes which meet in a point, and α, β, γ be the angles which a straight line through the point of intersection makes with each axis, the moment of the force round this line is $A\cos\alpha + B\cos\beta + C\cos\gamma$.

94. Three forces act on a point in directions respectively perpendicular to three rectangular co-ordinate planes, and each varying as the co-ordinate to which it is parallel; shew that there are two planes, in either of which if the point be situated the resolved part of the whole force, which is parallel to the plane, tends to the origin and varies as the distance of the point from it.

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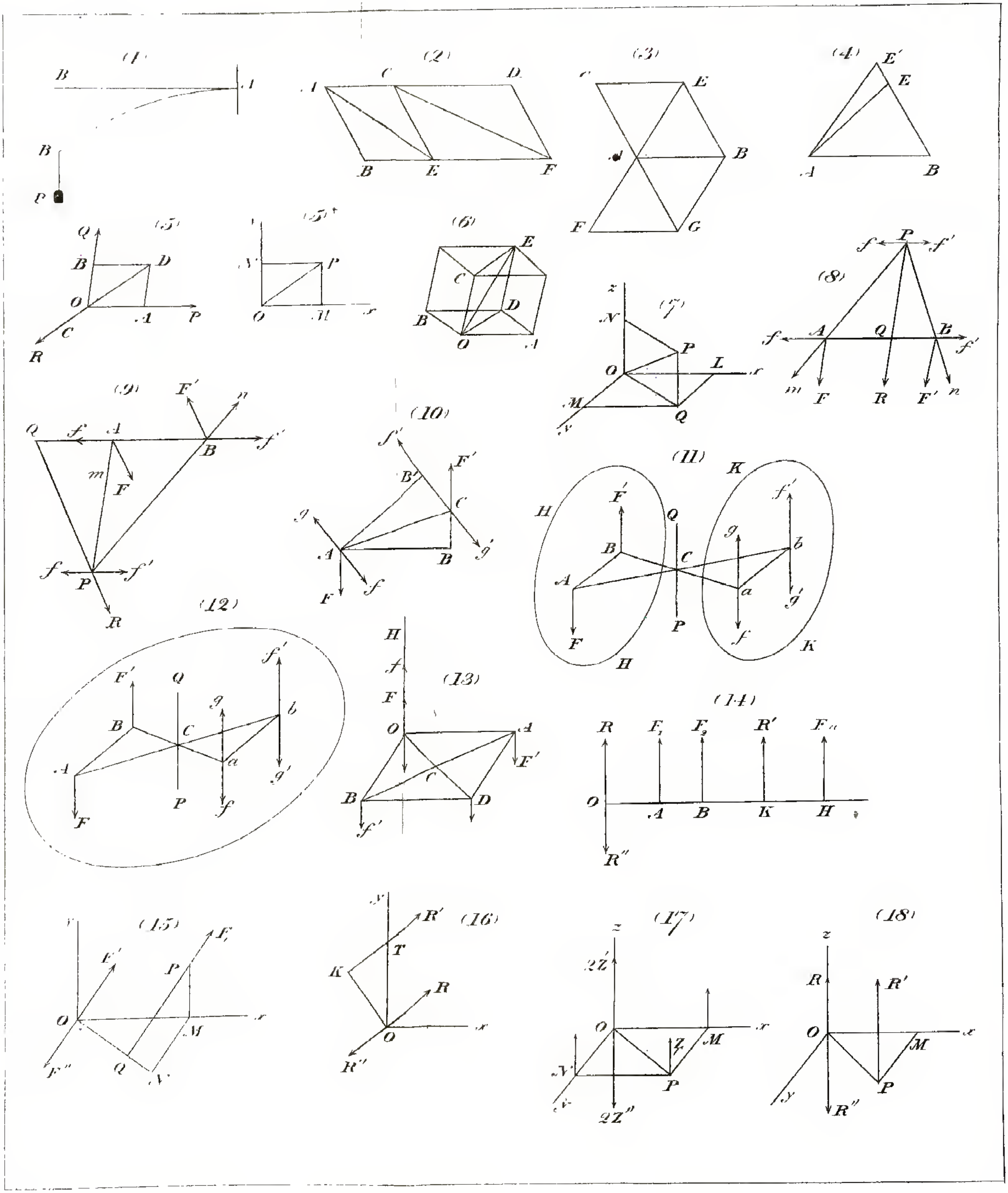
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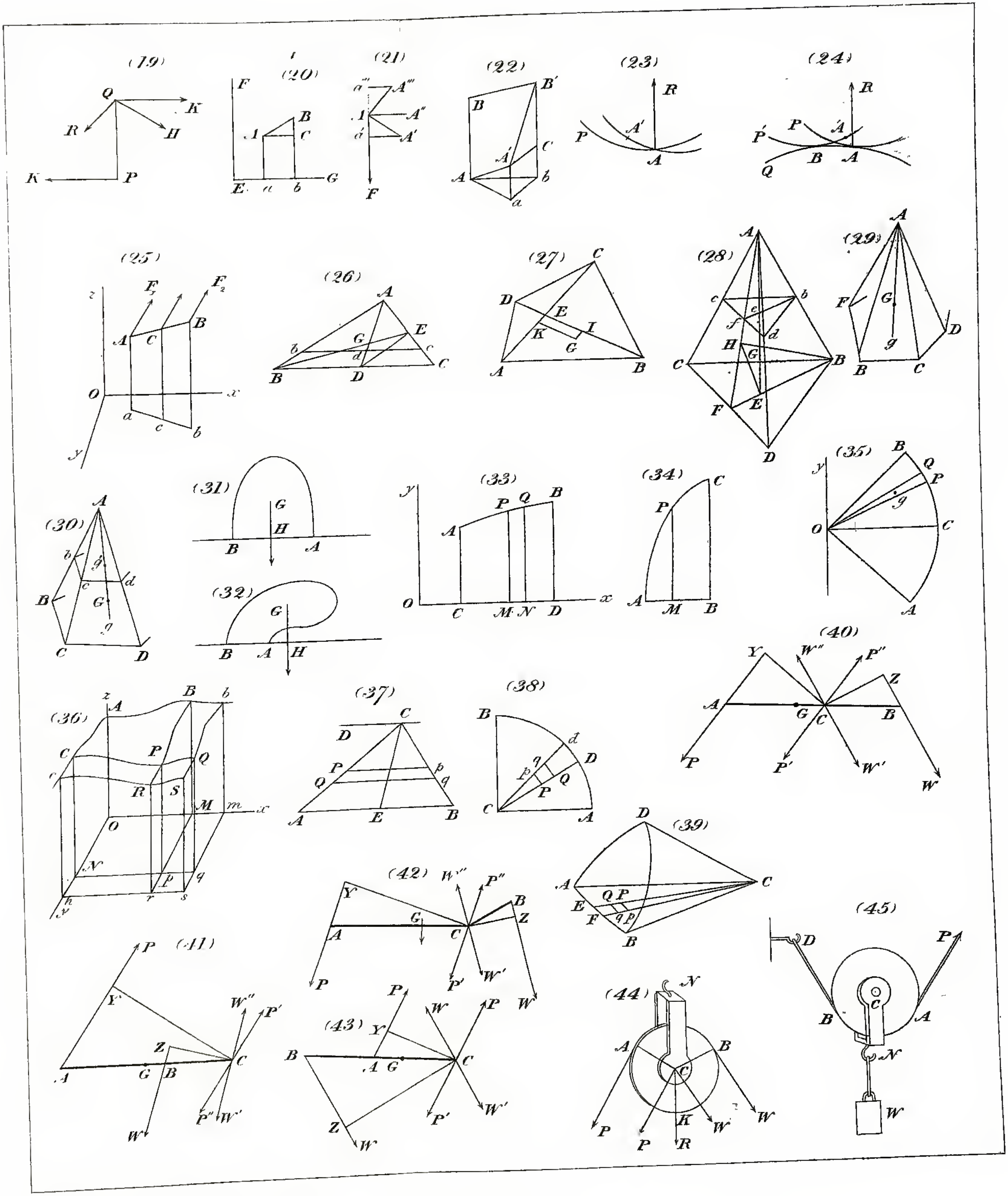
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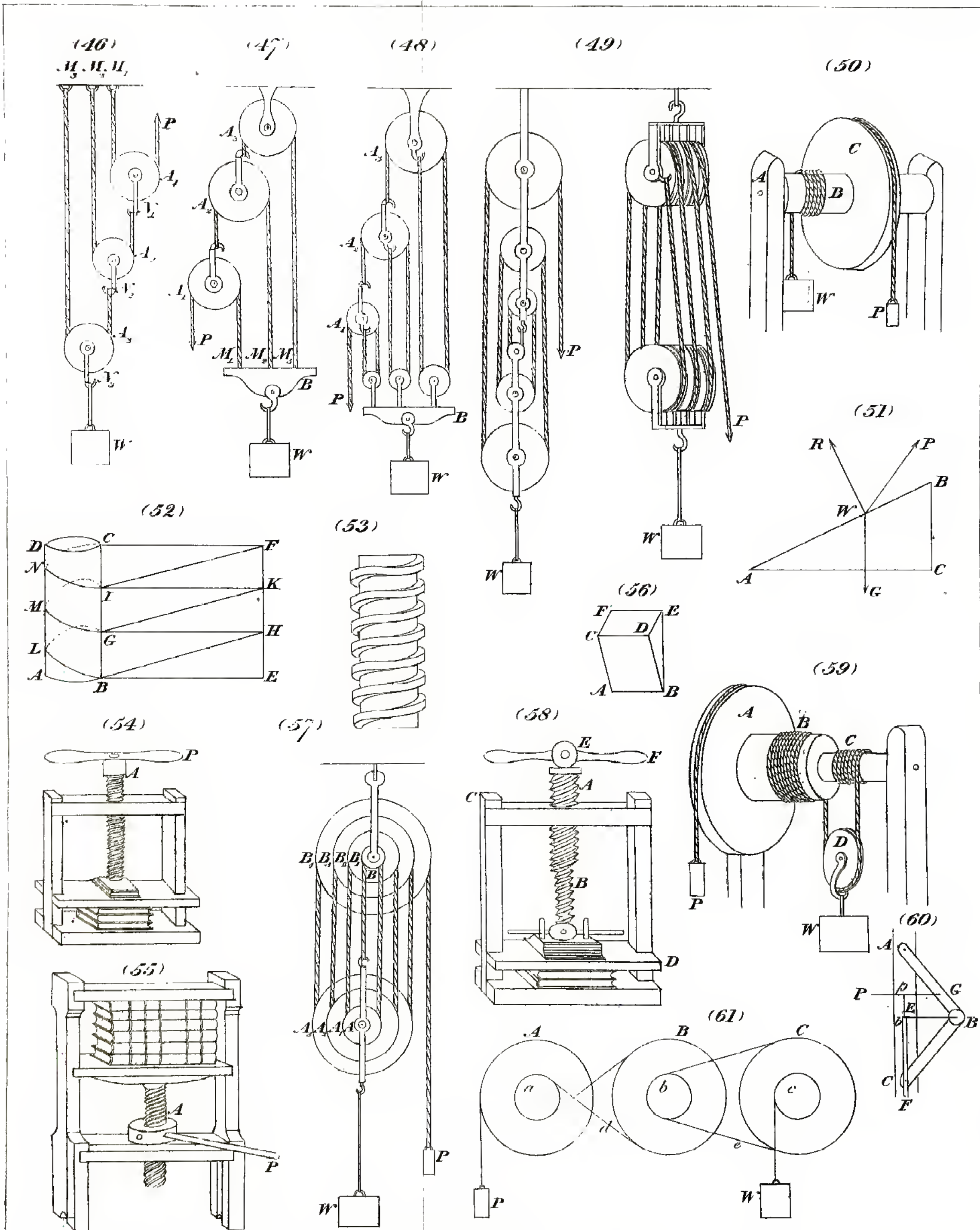
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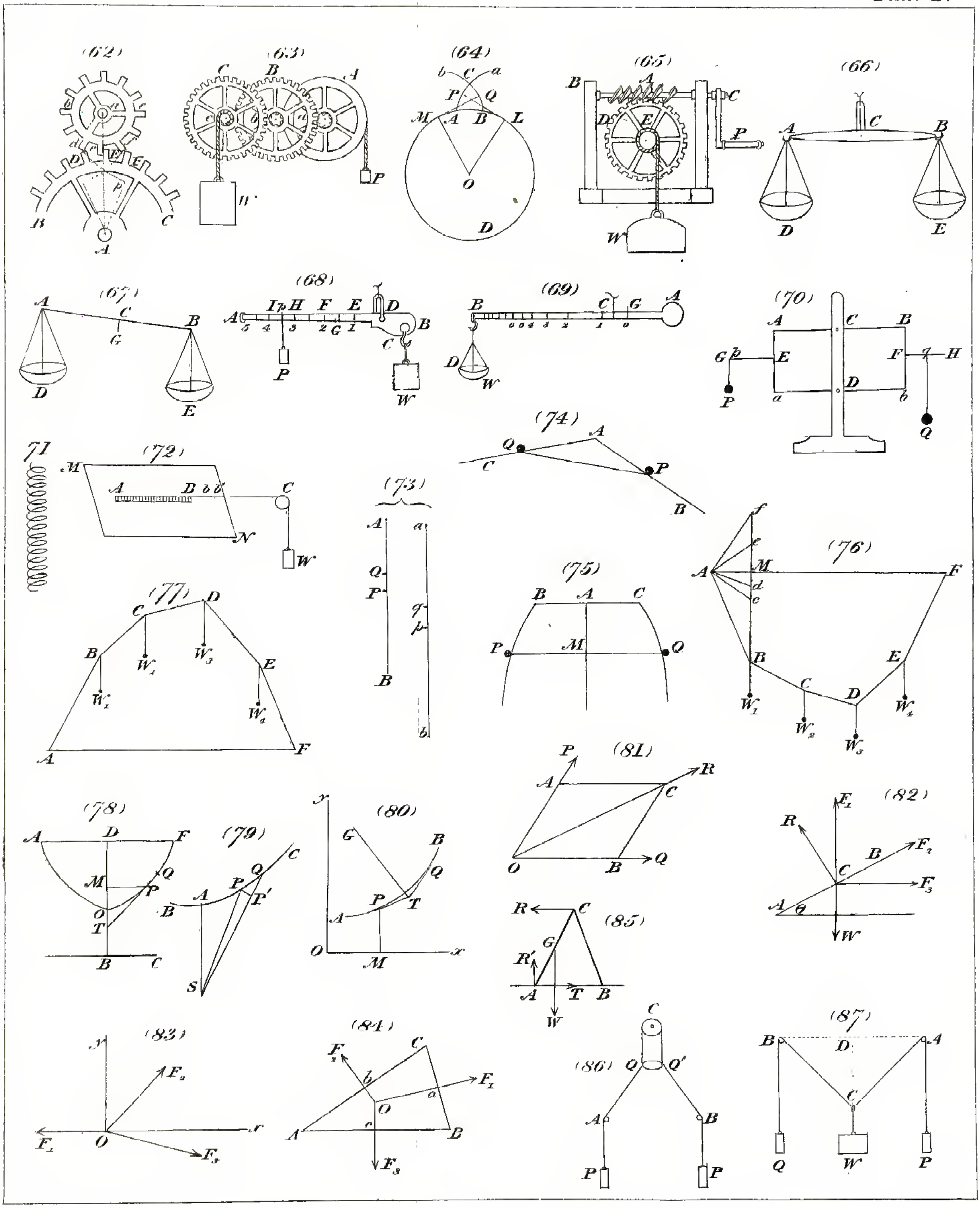
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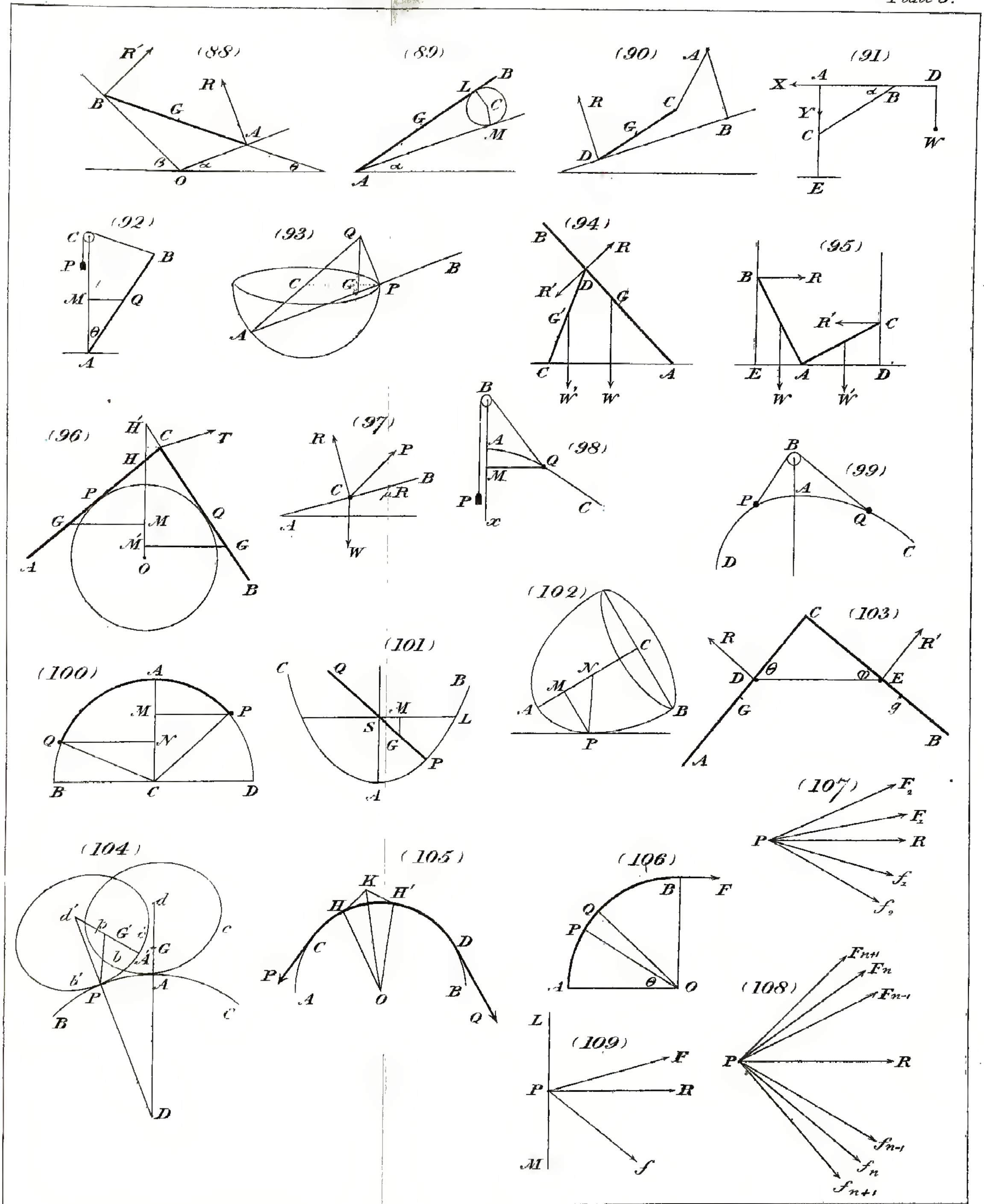
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